# PROCEEDINGS

OF THE

# NATIONAL ACADEMY OF SCIENCES, INDIA

1967

VOL. XXXVII | SECTION—A | PART I

NATIONAL ACADEMY OF SCIENCES, INDIA
ALLAHABAD

---

The *Proceedings of the National Academy of Sciences, India,* is published in two sections : Section A (Physical Sciences) and Section B (Biological Sciences). Four parts of each section are published annually (since 1960).

The Editorial Board in its work of examining papers received for publication is assisted, in an honorary capacity, by a large number of distinguished scientists. Papers are accepted from members of the Academy *in good standing*. In case of a joint paper, one of the authors must be a member of the Academy. The Academy assumes no responsibility for the statements and opinions advanced by the authors. The papers must conform strictly to the rules for publication of papers in the *Proceedings*. A total of 50 reprints is supplied free of cost to the author or authors. The authors may have any reasonable number of additional reprints at cost price, provided they give prior intimation while returning the proof.

Communications regarding contributions for publication in the *Proceedings*, books for review, subscriptions etc. should be sent to the General Secretary, National Academy of Sciences, India, Lajpatrai Road, Allahabad-2 (India).

**Annual Subscription for both Sections : Rs. 67 ; for each Section : Rs. 35 (including Annual Number) ; Single Copy : Rs. 7·50 ; Annual Number Rs. 5**

PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
INDIA
1967

VOL. XXXVII SECTION—A PART I

# A Theorem on Hankel Transform *†

By
P. N. RATHIE
*Department of Mathematics, M. R. Engineering College, Jaipur*

[Received on 30th December, 1964]

**Abstract**

The object of this paper is to establish a theorem on Hankel transform. The theorem is used to evalute certain infinite integrals involving products of Bessel and Gauss's hypergeometric functions. Some interesting particular cases are also mentioned.

**1. Introduction.** The well-known Hankel transform is defined as follows :

$$H_\nu\{f(t)\,;\,x\} = \int_0^\infty (xt)^{\frac{1}{2}} J_\nu(xt) f(t)\, dt,\ x > 0.$$

The aim of the present paper is to establish a theorem on the above transform and to use it to evaluate some integrals involving products of Bessel functions, Meijer's G–function and the Gauss's hypergeometric functions. A few interesting particular cases are also mentioned.

**2. Theorem.** If $p > 0$, $R(\alpha) > 0$, $R(\nu + \frac{1}{2}) > 0$, and the Hankel transform of $|f(t)|$ and $|\ t^{-\frac{1}{2}} \exp\left(-\frac{\alpha\, t^2}{4}\right) f(t^2/4)\ |$ exist, then

$$(1) \qquad p^{\frac{1}{2}} \int_0^\infty x^{\frac{1}{2}}(\alpha^2 + x^2)^{-\frac{1}{2}} \exp\left[-\frac{p^2\,\alpha}{\alpha^2 + x^2}\right] J_\nu\left[\frac{p^2\,x}{\alpha^2 + x^2}\right] H_\nu\{f(t)\,;\,x\}\, dx$$
$$= H_{2\nu}\{t^{-\frac{1}{2}} \exp(\frac{-\alpha t^2}{4})\, f(t^2/4)\,;\,p\}.$$

* Read at the 34th Annual Conference of the National Academy of Sciences, held at Muzaffarpur in February, 1965.

† Part of the Ph.D. thesis "A study of Bessel transforms" approved by the University of Jodhpur in 1965.

**Proof.** The left hand side of(1) is equal to

$$p^{\frac{1}{2}} \int_0^\infty x^{\frac{1}{2}} (\alpha^2 + x^2)^{-\frac{1}{2}} \exp\left[ -\frac{p^2 \alpha}{\alpha^2 + x^2} \right] J_\nu \left[ \frac{p^2 x}{\alpha^2 + x^2} \right]$$

$$\left[ \int_0^\infty (tx)^{\frac{1}{2}} J_\nu(tx) f(t)\, dt \right] dx$$

$$= p^{\frac{1}{2}} \int_0^\infty t^{\frac{1}{2}} f(t) \left[ \int_0^\infty x(\alpha^2 + x^2)^{-\frac{1}{2}} \exp\left[ -\frac{p^2 \alpha}{\alpha^2 + x^2} \right] \right.$$

$$\left. J_\nu(tx) J_\nu \left[ \frac{p^2 x}{\alpha^2 + x^2} \right] dx \right] dt;$$

on interchanging the order of integration.

Evaluating the $x$ - integral by [2, p. 58(14)], this becomes

$$p^{\frac{1}{2}} \int_0^\infty t^{-1/2} e^{-\alpha t} J_{2\nu}(2pt^{1/2}) f(t)\, dt$$

$$= H_{2\nu}\{t^{-1/2} \exp\ (-\alpha t^2/4) f(t^2/4) ; p\},$$

on changing $t$ to $t^2/4$.

The change of the order of integration is justified when :

(*i*) the Hankel transform of $|f(t)|$ exists,

(*ii*) the Hankel transform of $|t^{-\frac{1}{2}} \exp(-\alpha t^2/4) f(t^2/4)|$ exists,

and (*iii*) the $x$ - integral is absolutely convergent, *i.e.*, when $R(\alpha) > 0$ and $R(\nu + \frac{1}{2}) > 0$.

3. **Applications.** In this section the theorem is illustrated by suitable examples.

**Example 1.** If we take

$$f(t) = t^{\sigma - 3/2} e^{-\beta t},$$

then [2, p. 29(7)]

$$H_\nu\{t^{\sigma - 3/2} e^{-\beta t} ; x\} = 2^{-\nu} \beta^{-\sigma-\nu} x^{\nu+\frac{1}{2}} \Gamma(\nu + \sigma)$$

$$[\Gamma(\nu+1)]^{-1} {}_2F_1[\sigma/2 + \nu/2, \sigma/2 + \nu/2 + \tfrac{1}{2} ; \nu + 1 ; -x^2/\beta^2],$$

for $R(\sigma + \nu) > 0, R(\beta) > 0, x > 0;$

and [2, p. 30 (14)]

$$H_{2\nu}\{3^{3-2\sigma} t^{2\sigma - 7/2} \exp - (\alpha + \beta) t^2/4 ; p\}$$

$$= p^{2\nu + \frac{1}{2}} (\alpha + \beta)^{-\nu-\sigma+1} \Gamma(\nu + \sigma - 1) [\Gamma(2\nu + 1)]^{-1}$$

$${}_1F_1[\nu + \sigma - 1 ; 2\nu + 1 ; -p^2/(\alpha + \beta)],$$

for $R(\alpha + \beta) > 0, R(\nu + \sigma - 1) > 0, p > 0.$

Hence (1) yields

$$(2)\qquad \int_0^\infty x^{\nu+1}(\alpha^2+x^2)^{-\frac{1}{2}} \exp\left[-\frac{p^2\alpha}{\alpha^2+x^2}\right] I_\nu\left[\frac{p^2x}{\alpha^2+x^2}\right]$$

$$ {}_2F_1[\sigma/2+\nu/2, \sigma/2+\nu/2+1/2\,;\,\nu+1\,;\,-x^2/\beta^2]\,dx$$

$$= 2^\nu\,\beta^{\sigma+\nu}\,p^{2\nu}\,(\alpha+\beta)^{-\sigma-\nu+1}\,\Gamma(\nu+1)\,(\sigma+\nu-1)^{-1}$$

$$[\Gamma(2\nu+1)]^{-1}\,{}_1F_1[\nu+\sigma-1\,;\,2\nu+1\,;\,-p^2/(\alpha+\beta)]\,;$$

for. $\qquad R(\alpha)>0,\ R(\nu+1)>0,\ R(\nu+\sigma-1)>0,\ R(\beta)>0,\ p>0.$

As $\alpha\to 0$, (2) reduces to a particular case of [2, p. 82(9)].

When $\sigma=\nu+2$, (2) reduces to

$$(3)\qquad \int_0^\infty x^{\nu+1}(\alpha^2+x^2)^{-\frac{1}{2}}(\beta^2+x^2)^{-\nu-3/2}\exp\left[-\frac{p^2\alpha}{\alpha^2+x^2}\right] J_\nu\left[\frac{p^2\,x}{\alpha^2+x^2}\right]dx$$

$$= \pi^{\frac{1}{2}}\,p^{2\nu}\,(\alpha+\beta)^{-2\nu-1}\,2^{-\nu-1}\,\beta^{-1}\,[\Gamma(\nu+3/2)]^{-1}\exp\left[-p^2/(\alpha+\beta)\right]$$

for $\qquad R(\alpha)>0,\ R(\beta)>0,\ R(\nu+1)>0,\ p>0.$

**Example 2.** Taking

$$f(t) = t^{\lambda/2+1/4}\,K_\mu(\alpha\, t),$$

the theorem gives the following result on using [2, p. 63(4)] and [2, p. 69(15)].

$$(4)\qquad \int_0^\infty x^{\nu+1}(\alpha^2+x^2)^{-\frac{1}{2}}\exp\left[-\frac{p^2\,\alpha}{\alpha^2+x^2}\right]J_\nu\left[\frac{p^2\,x}{\alpha^2+x^2}\right]$$

$${}_2F_1[\tfrac{1}{2}(\nu-\mu+\lambda/2+7/4),\ \tfrac{1}{2}(\nu+\mu+\lambda/2+7/4)\,;\,\nu+1\,;\,-x^2/\alpha^2]\,dx$$

$$= \pi^{\frac{1}{2}}\,\alpha^{\nu+\lambda/2+7/4}\,p^{-\lambda-3/2}\,\Gamma(\nu+1)\,2^{-\lambda/2+1/4}\,[\Gamma\{\tfrac{1}{2}(\nu\pm\mu+\lambda/2+7/4)\}]^{-1}$$

$$G^{1,2}_{2,3}\left[\frac{p^2}{2\alpha}\,\middle|\,\begin{matrix}1-\mu\,,\,1+\mu\\ 3/4+\lambda/2+\nu\,,\,\tfrac{1}{2},\,3/4+\lambda/2-\nu\end{matrix}\right],$$

for $\qquad p>0,\ R(\alpha)>0,\ R(\nu+1)>0,\ R(\lambda+2\nu\pm 2\mu+3/2)>0.$

When $\mu=\lambda/2-\nu-1/4$, (4) reduces to

$$(5)\qquad \int_0^\infty x^{\nu+1}(\alpha^2+x^2)^{-\lambda/2-5/4}\exp\left[-\frac{p^2\alpha}{\alpha^2+x^2}\right]J_\nu\left[\frac{p^2x}{\alpha^2+x^2}\right]dx$$

$$= 2^{-\lambda-\nu-\frac{1}{2}}\,p^{2\nu}\,\pi^{\frac{1}{2}}\,\alpha^{-\lambda-\frac{1}{2}}\,[\Gamma(\lambda/2+3/4)\,\Gamma(\lambda/2+\nu+5/4)]^{-1}\,\Gamma(\lambda+\tfrac{1}{2})$$

$${}_1F_1[\lambda+\tfrac{1}{2}\,;\,\nu+\lambda/2+5/4\,;\,-p^2/(2\,\alpha)]\,,$$

for $\qquad R(\alpha)>0,\ p>0,\ R(\lambda+\tfrac{1}{2})>0,\ R(\nu+1)>0.$

As $p\to 0$, (5) gives a known result [1, p. 310 (19)].

**Example 3.** Taking

$$f(t) = (4t)^{\lambda/2+\frac{1}{4}}\,J_\mu(2\,\beta\,t^{\frac{1}{2}}),$$

using [1, p. 187 (43)]; the theorem yields the following result :

$$(6)\qquad \int_0^\infty x(\alpha^2+x^2)^{-\frac{1}{2}} \exp\left[-\frac{p^2\alpha}{\alpha^2+x^2}\right] J_\nu\left[\frac{p^2 x}{\alpha^2+x^2}\right]$$

$$G^{2,1}_{2,4}\left[\frac{\beta^4}{4x^2}\middle|\begin{matrix}1-\nu/2,\ 1+\nu/2\\ \triangle(2\,;\,\lambda/2+\mu/2+2),\ \triangle(2\,;\,\lambda/2-\mu/2+2)\end{matrix}\right] dx$$

$$= 2^{-\lambda-3}\,\alpha^{-\lambda/2-\mu/2-\nu-1}\,\beta^{\lambda+\mu+4}\,p^{2\nu}[\Gamma(\mu+1)\,\Gamma(2\nu+1)]^{-1}$$

$\Gamma(\nu+1+\lambda/2+\mu/2)\ \ \psi_2(\nu+1+\lambda/2+\mu/2\,;\,\mu+1,\,2\nu+1\,;\,\beta^2/\alpha,\,p^2/\alpha)$,

for $\quad R(\alpha)>0,\ R(\lambda/2+\mu/2+\nu+1)>0,\ R(\nu+1)>0,\ p>0,\ \beta>0.$

**Example 4.** Lastly, taking

$$f(t) = t^{-\frac{1}{2}}\,e^{-\beta t}\,J_{2\nu}(2\,\gamma\,t^{\frac{1}{2}}),$$

the theorem gives the following result on using [2, p. 58(17)] and [1, p. 187(43)].

$$(7)\qquad \int_0^\infty x(\alpha^2+x^2)^{-\frac{1}{2}}\ (\beta^2+x^2)^{-\frac{1}{2}}\ \exp\left[-\frac{p^2\alpha}{\alpha^2+x^2}\right]$$

$$\exp\left[-\frac{\gamma^2\beta}{\beta^2+x^2}\right] J_\nu\left[\frac{p^2x}{\alpha^2+x^2}\right] J_\nu\left[\frac{\gamma^2x}{\beta^2+x^2}\right] dx$$

$$=(\gamma\,p)^{2\nu}\,(\alpha+\beta)^{-2\nu}\,\Gamma(2\nu)\,[\Gamma(2\nu+1)]^{-2}$$

$$\psi_2[2\nu\,;\,2\nu+1,\,2\nu+1\,;\,\gamma^2/(\alpha+\beta),\,p^2/(\alpha+\beta)],$$

for $\quad R(\alpha)>0,\ R(\beta)>0,\ R(\nu)>0.$

When $\quad \gamma=0$, (7) reduces to

$$(8)\qquad \int_0^\infty x^{\nu+1}\,(x^2+x^2)^{-\frac{1}{2}}\,(\beta^2+x^2)^{-\nu-\frac{1}{2}}\exp\left[-\frac{p^2\alpha}{\alpha^2+x^2}\right] J_\nu\left[\frac{p^2x}{\alpha^2+x^2}\right] dx$$

$$=2^\nu\,p^{2\nu}\,(\alpha+\beta)^{-2\nu}\,\Gamma(2\nu)\,\Gamma(\nu+1)\,[\Gamma(2\ \ +1)]^{-2}\,{}_1F_1\left[2\nu\,;\,2\nu+1\,;\,\frac{p^2}{\alpha+\beta}\right],$$

for $\quad R(\alpha)>0,\ R(\beta)>0,\ R(\nu)>0.$

Similar results can be easily obtained by taking $\beta\to 0$ and $p\to 0$ in result (7); the last one gives a known result [1, p. 310(19)].

**References**


1. Erdelyi, A. Tables of integral transforms. *McGraw-Hill, New York*, **Vol. I** : (1954)..
2. Erdelyi, A Tables of integral transform. *McGraw-Hill, New York*, **Vol. II** : (1954).

# Kinetics of the Oxidation of Glycerol by Hexavalent Chromium

*By*

K. S. SRIVASTAVA *and* S. GHOSH

*Chemical Laboratories, University of Allahabad, Allahabad*
*and Department of Post Graduate Studies and Research in Chemistry, Jabalpur University*

[Received on 30th April, 1965]

**Abstract**

Oxidation of alcohols, both primary and secondary, by chromic acid have been investigated by several workers. In this paper kinetics of the oxidation of polyhydric alcohol glycerol by potassium dichromate in the presence of sulphuric acid has been investigated. The first order constants with respect to oxidant have been found to increase with the progress of the reaction, excepting at lower temperature and at lower concentration of sulphuric acid. At higher concentrations of sulphuric acid the reaction tends to to be of zero order with respect to the oxidant. The order with respect to glycerol as well as with respect to sulphuric acid have been found to be one. Thus the total order of the reaction is three, which decreases to two with increasing concentration of sulphuric acid and temperature. A mechanism for the oxidation of Glycerol by Hexavalent Chromium has been suggested.

Glycerine is completely oxidised to carbon-dioxide and water when refluxed with chromic acid on a water bath in the presence of an excess of sulphuric acid. It is, however, certain that the oxidation of glycerol occurs in several steps and several intermediatary products are likely to be formed during its oxidation. The extent of oxidation under different conditions may be determined from the equivalent number, but the identification of the intermediatary products in the reaction is not easy. Sulphuric acid is well known to be a catalyst for the oxidations effected by chromic acid and here we have investigated the kinetics of the oxidation of glycerol by hexavalent chromium in the presence of different amount of sulphuric acid at different temperatures.

## Experimental

The chemicals used were of A. R. quality. Glycerol was estimated quantitatively by refluxing a measured volume of its aqueous solution with chromic acid and sulphuric acid mixture in a flask for about two hours. The unused chromic acid was estimated iodometrically.

A reaction bottle of jena glass was coated outside with black japan to avoid any effect of light and the reaction was carried out in a precision thermostat. To a definite volume of glycerol solution and sulphuric acid a measured volume of potassium dichromate solution was added and the progress of the reaction was studied by estimating the unused oxidant iodometrically.

The values of the first order constants with respect to the oxidant were calculated for different times. As the constants under different conditions of experiment varied beyond the experimental error, it is necessary to reproduce some of the exprimental data in the following tables :

TABLE 1

*Final concentrations of the reactants*

*Glycerol* 0·125N ; *Potassium dichromate* 0·02N ; *Sulphuric acid* 0·1N

| Time in Mts. | $K25° \times 10^4$ | $K35° \times 10^4$ | $K45° \times 10^4$ |
|---|---|---|---|
| 0 | – | – | – |
| 15 | – | 5·87 | 10·7 |
| 30 | 3·56 | 5·93 | 10·8 |
| 60 | 3·26 | 6·03 | 10·4 |
| 90 | 3·20 | 6·16 | 10·8 |
| 120 | 3·18 | 6·14 | 11·1 |
| 150 | 3·05 | 6·17 | 11·3 |
| 180 | 2·98 | 6·23 | 11·5 |
| Mean value | 3·21 | 6·08 | 10·94 |

The reaction between glycerol and dichromate is slow at 25°C or below but in the presence of 0·1N sulphuric acid the reaction proceeds sufficiently fast at 25°C.

**Effect of sulphuric acid**

For studying the effect of sulphuric acid on the reaction rate, experiments were performed with different concentrations of sulphuric acid. Results for 0·2N, 0·4N, 0·8N sulphuric acid at 45°C are given in table 2. The observations with 0·1N sulphuric acid at 45°C has been already given in table 1 column 4.

TABLE 2

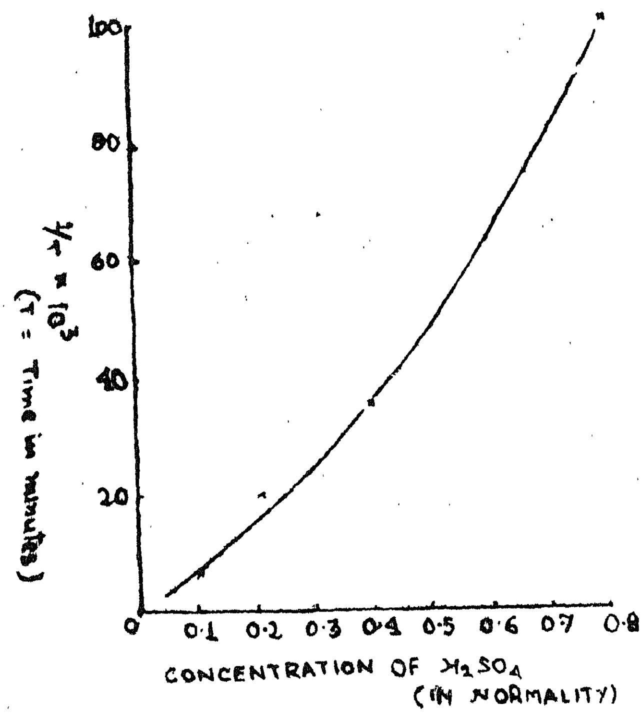
| Time in Mts. | $K \times 10^4$ for 0·2N $H_2SO_4$ | $K \times 10^4$ for 0·4N $H_2SO_4$ | $K \times 10^4$ for 0·8N $H_2SO_4$ |
|---|---|---|---|
| 0 | – | – | – |
| 10 | 28·8 | 88·2 | 304·5 |
| 15 | – | – | 326·3 |
| 20 | 29·7 | 98·3 | 341·0 |
| 30 | 30·9 | 108·8 | 357·1 |
| 40 | 32·9 | 117·1 | – |
| 50 | 34·3 | 123·9 | – |
| 60 | 35·5 | – | – |

Times for the half decomposition of the oxidant with different concentration of sulphuric acid at 45°C has been graphically obtained and are noted below :

TABLE 3

| Concentration of $H_3SO_4$ in the reaction mixture | Time for the half decomposition of oxidant |
|---|---|
| 0·1N | 145 min |
| 0·2N | 79 ,, |
| 0·4N | 28·5 ,, |
| 0·8N | 10·0 ,, |

It will be seen in Table 1 and 2 that the first order constants tend to decrease at lower temperature and at lower concentration of $H_2SO_4$ while reverse is true for higher temperatures and higher concentrations of sulphuric acid.




The effect of the concentration of sulphuric acid on the reaction velocity is represented in fig. 1 where the times for half decomposition of the oxidant verses inverse of the concentration of sulphuric acid have been plotted. It will be seen that the time of half decomposition is proportional to the inverse of concentration of sulphuric acid. Hence we may conclude that

$$\frac{1}{2} \times C\,(H_2SO_4) - \text{const.}$$

It should be also noted here that as the concentration of $H_2SO_4$ increases the first order rate constants rapidly increase, espceailly at higher temperatures. In other words as the concentration of $H_2SO_4$ or the temperature is increased, the reaction tends to be of fractional or zero order with respect to the oxidant.

Temperature coefficient and heat of activations have been calculated from table 1 by taking the average value and are given in the following table :

TABLE 4

| Temperaturue range | Concentration of sulphuric acid | Temperature coefficient | Heat of activation |
|---|---|---|---|
| 25°C - 30°C | 0·1N | 1·9 | 11490 cal. |
| 35°C - 45°C | 0·1N | 1·8 | 11240 cal. |

**Total order of the reaction**

The order of the reaction with respect to glycerol has been determined by estimating the loss of oxidant, for the reaction mixture - containing the concentration of Glycerol double of that used in earlier experiments and then applying the relation

$$n = \frac{\log K' - \log K}{\log 2.}$$

Total order of the reaction has been calculated from that data where 0·1N sulphuric acid is used and the temperature of the reaction is 25°C. The reaction has been found to be approximately of first order with respect to the oxidant and with respect to glycerol its value is 1·2. In other words the total order of the reaction is approximately two.

**Effect of bivalent Manganese**

Bivalent manganese is known to catalyse reaction between oxalic acid and hexavalent chromium[1] in acid medium whilst it has been noted by Westheimer[2] *et al.* and Chatterji and coworkers[3] that its oxidation by hexavalent chromium is induced by an alcohol. We have also found that in presence of bivalent manganese, the acidified reaction mixture of glycerol and potassium. dichromate develops asred colour due to the formation of colloidal $MnO_2$ and the rate of the total loss of the oxidant by glycerol in presence of bivalent manganese is also decreased.

**Conclusion**

The oxidation of alcohols by chromic acid have been closely investigated by several workers[4,5,6] and in case of alcohols Westheimer *et al.*, have suggested an intermediate formation of an ester with chromic acid and its subsequent decomposition leading to the oxidised products of the alcohol and lower valency states of chromium.

Glycerol having three hydroxy groups is known to form ester with all the three -OH groups. Accordingly the three moles of chromic acid are likely to

enter into ester formation, which may undergo electron transfer leading to the formation of chromium of lower valency and the oxidised product of glycerol. If the rate determining step is the esterification, then the rate of oxidation is likely to be of order three with respect to the oxidant, which is not observed in the present study. Therefore, the ester leading to oxidation of glycerol involves one - OH group at a time, possibly the oxination at the primary position being easier occurs first.

In the oxidation effected by chromic acid, the anion $HCrO_4^{-1}$ is reduced to $Cr^{+3}$ and this will always involve hydrogen ions in the reaction thus :

$$HCrO_4^{-1} + 4H^+ \rightarrow Cr^{+3} + 2\tfrac{1}{2}\ H_2O + 1\tfrac{1}{2}\ O$$

Moreover, hydrogen ions are well known to catalyse ester formation and therefore the utility of hydrogen ions is necessary in the oxidation by chromic acid and our experimental results also establish this fact.

It should be mentioned here that whenever the hydrogen ion concentration or the temperature is increased the first order constants increases with the progress of the reaction, this is possible if the intermediatary oxidised products of glycerol consume the oxidant at a faster rate.

The oxidation of OH group in the primary positions on oxidation will yield aldehyde groups. But we have generally noted that the oxidation of either $CH_3CHO$ or HCHO by Hexavalent chromium ($Cr^{+6}$) required sufficiently large quantity of sulphuric acid for their oxidation. However, it is not unlikely that a - CHO group is easily oxidised when attached with a CHOH group as we have noted that even a COOH can be oxidised when attached with CHOH group as in lactic acid.

Thus one of the Intermediate products of the oxidation is COOH CHOH COOH. The further oxidation of which will lead to the formation of meso-oxalic acid. Our measurements of equivalents of chromic acid required to oxidise one molecule of glycerol at the temperatures of our experiments ranges between 10 to 12 indicating that the oxidation of organic substrate is not complete as the equivalent no. should be 14. Moreover, we have observed the p esence of HCOOH as one of the oxidised products. In view of these observations we conclude that the meso-oxalic acid decomposes of itself as :

$$\begin{array}{l} COOH \\ \quad \diagdown \\ \quad\ CO \rightarrow HCHO + 2CO_2 \\ \quad \diagup \\ COOH \end{array}$$

The HCHO at high concentration of acids and the oxidant at high temperatures will lead to the formation of formic acid.

Thus it is clear in the oxidation of glycerol by $Cr^{+6}$, oxalic acid cannot be an intermediate oxidation product because at the concentration of reactant used in the experiment oxalic acid will be completely oxidised and therefore the equivalent number should be fourteen.

We have observed that a carboxylic group attached with a carbon atom carrying either -OH *viz.*, $CH_3CHOHCOOH$ or = O *viz.*, $CH-\overset{\overset{O}{\|}}{C}-COOH$ or -O -OH *viz.*, COOH - COOH is easily oxidisable even with lower concentrations of sulphuric acid and hence the reaction becomes fast.

**Acknowledgement**

Authors express their thanks to the Council of Scientific and Industrial Research, New Delhi, for financial assistance.

**References**

1. Ghosh and Chakravarty.
2. Westheimer and Novic. *J. Chem. Phys.*, **11** : 500, (1943).
3. Chat erji and Antony. *Proc. Nat. Acad. Sci.*, (India) **23A** : 20, (1954).
4. Slack and Waters. *J. Chem. Soc.*, **70** : 1666, (1948) ; *J. Chem. Soc*, 599, (1949) ; *J Chem Soc*, 594, (1949).
5. Mosher and Whitmore. *J. Chem. Soc.*, **70** : 2544, (1948).
6. Mosher and Langerak. *Ibid*, **71** : 286, (1949) ; **73** : 1302, (1951).
7. Hampton, Leo and Westheimer. *J Chem. Soc*, **78** : 306, (1956).

# Some Properties of $M_{k,m}$ Transform

*By*

S. K. KULSHRESHTHA*

*Department of Mathematics, University of Rajasthan, Jaipur*

[Received on 10th February, 1966]

**Abstract**

In this paper some properties of $M_{k,m}$ transform are obtained. Using these properties some recurrence relations for Gauss's hypergeometric functions are also obtained. Some of the theorems obtained in this paper are analogous to the theorems for Whittaker transform. Some infinite integrals involving Whittaker function are also evaluated with the help of the theorems.

**1. Introduction.** I[1] have generalised the well known Laplace transform defined by the integral equation

$$\phi(p) = p\int_0^\infty e^{-pt}\, h(t)\, dt \tag{1·1}$$

in the form

$$\psi(p) = p\int_0^\infty (2pt)^{-\frac{1}{2}}\, M_{k,m}\,(2pt)\, h(t)\, dt. \tag{1·2}$$

(1·2) reduces to (1·1), when we take $k = -m = 1/4$, due to the identity

$$M_{\frac{1}{4},-\frac{1}{4}}(x) \equiv x^{\frac{1}{4}}\, e^{-x/2} \tag{1·3}$$

The $M_{k,m}$ transform (1·2) can also be related with Whittaker transform given by Varma[3] due to the identity

$$W_{k,m}(x) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)}\, M_{k,m}(x) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)}\, M_{k,-m}(x) \tag{1·4}$$

We shall denote $M_{k,m}$ transform (1·2) symbolically as

$$\phi(p) \;\underset{k,m}{\overset{M}{=\!=}}\; h(t)$$

and (1·1) as usual shall be denoted as

$$\psi(p) \doteqdot h(t).$$

**2. Theorem 1.** The recurrence formula that holds for $M_{k,m}(Z)$, also holds for $M_{k,m}$ transform of the function $x^{-\lambda}\, h(x)$, where $\lambda$ is any arbitrary parameter, provided the integrals and the series involved are convergent.

**Present address*: Department of Mathematics, Samrat Ashok Technological Institute. Vidisha (M. P.), India.

**Proof.** We have the result [2, p. 27]

$$M_{k,m}(Z) = Z^{\frac{1}{2}} M_{k-\frac{1}{2},m-\frac{1}{2}}(Z) - \frac{(\frac{1}{2}-k+m)}{2m(2m+1)} M_{k-\frac{1}{2},m+\frac{1}{2}}(Z).$$

Let

$$\phi_{k,m,\lambda}(p) \underset{k,m}{\overset{M}{=}} x^{-\lambda} h(x)$$

that is

$$\phi_{k,m,\lambda}(p) = p \int_0^{\infty} (2px)^{-\frac{1}{2}} M_{k,m}(2px)\, x^{-\lambda} h(x)\, dx \tag{2·1}$$

$$= p \int_0^{\infty} x^{-\lambda} (2px)^{-\frac{1}{2}} h(x) \left[ (2px)^{\frac{1}{2}} M_{k-\frac{1}{2},m-\frac{1}{2}}(2px) - \frac{(\frac{1}{2}-k+m)}{2m(2m+1)} (2px)^{\frac{1}{2}} M_{k-\frac{1}{2},m+\frac{1}{2}}(2px) \right] dx$$

Hence, we get the recurrence formula

$$\frac{1}{\sqrt{2p}} \phi_{k,m,\lambda}(p) = \phi_{k-\frac{1}{2},m-\frac{1}{2},\lambda-\frac{1}{2}}(p) - \frac{(\frac{1}{2}-k+m)}{2m(2m+1)} \phi_{k-\frac{1}{2},m+\frac{1}{2},\lambda-\frac{1}{2}}(p), \tag{2·2}$$

provided the integral involved in the result (2·1) is absolutely convergent.

The integral in (2·1) is uniformly and absolutely convergent if $R(\rho - \lambda + m + 5/4) > 0$ where $h(x) = 0\,(x^{\rho})$ for small $x$ and $R(p) \geqq R(p_0) > 0$.

Similarly, using the other recurrence formulae for $M_{k,m}(Z)$ [2, p. 27], we get the following recurrence formulae for the $M_{k,m}$ transform as following :

(i) $$\frac{1}{\sqrt{2p}} \phi_{k,m,\lambda}(p) = 2m\, \phi_{k-\frac{1}{2},m-\frac{1}{2},\lambda+\frac{1}{2}}(p) - 2m\, \phi_{k+\frac{1}{2},m-\frac{1}{2},\lambda+\frac{1}{2}}(p), \tag{2·3}$$

(ii) $$\frac{(2m+1)}{\sqrt{2p}} \phi_{k,m,\lambda}(p) = (\tfrac{1}{2}-k+m)\, \phi_{k-\frac{1}{2},m+\frac{1}{2},\lambda+\frac{1}{2}}(p) + \frac{(\frac{1}{2}+k+m)}{2m} \times \phi_{k+\frac{1}{2},m+\frac{1}{2},\lambda+\frac{1}{2}}(p), \tag{2·4}$$

(iii) $$\sqrt{2p}\, \phi_{k,m,\lambda}(p) = \phi_{k+\frac{1}{2},m+\frac{1}{2},\lambda+\frac{1}{2}}(p) + \frac{(\frac{1}{2}+k+m)}{2m(2m+1)} \phi_{k+\frac{1}{2},m+\frac{1}{2},\lambda+\frac{1}{2}}(p) \tag{2·5}$$

(iv) $$2m\, \phi_{k,m,\lambda}(p) - 2p\, \phi_{k,m,\lambda-1}(p) = 2m\sqrt{2p}.\ \phi_{k+\frac{1}{2},m-\frac{1}{2},\lambda-\frac{1}{2}}(p) - \frac{(\frac{1}{2}-k+m)}{2m+1} \sqrt{2p}\, \phi_{k-\frac{1}{2},m+\frac{1}{2},\lambda-\frac{1}{2}}(p) \tag{2·6}$$

(v) $$2m\, \phi_{k,m,\lambda}(p) - 2p\, \phi_{k,m,\lambda-1}'(p) = \sqrt{2p}\, \frac{(\frac{1}{2}+k+m)}{2m+1} \phi_{k+\frac{1}{2},m+\frac{1}{2},\lambda-\frac{1}{2}}(p) - 2m\sqrt{2p}.\ \phi_{k-\frac{1}{2},m-\frac{1}{2},\lambda-\frac{1}{2}}(p) \tag{2·7}$$

and (vi) $$2k\, \phi_{k,m,\lambda}(p) - 2p\, \phi_{k,m,\lambda-1}(p) = (\tfrac{1}{2}+k+m)\, \phi_{k+1,m,\lambda}(p) + (\tfrac{1}{2}-k+m)\, \phi_{k-1,m,\lambda}(p) \tag{2·8}$$

**Example.** We have the result

$$x^{-\lambda}h(x) = x^{-\lambda}\, e^{-a x}$$

$$\underset{k,m}{\overset{M}{=}} \quad \frac{p(2p)^{m+\frac{1}{2}}\,\Gamma(m-\lambda+5/4)}{(\alpha+p)^{m-\lambda+5/4}}\; {}_2F_1\left[\begin{matrix}\tfrac{1}{2}-k+m,\; m-\lambda+5/4 \\ 2m+1\end{matrix};\; \frac{2p}{(\alpha+p)}\right] \tag{2·9}$$

where $R(m-\lambda+5/4) > 0$ and $R(\alpha) > R(p) > 0$,

then using the result (2·9) in the result (2·2) and adjusting the parameters, we get the recurrence formula for the Gauss's hypergeometric function as

$${}_2F_1\left[\begin{matrix}\alpha,\ \beta \\ \gamma+1\end{matrix};\ x\right] - {}_2F_1\left[\begin{matrix}\alpha,\ \beta \\ \gamma\end{matrix};\ x\right] + \frac{\alpha\,\beta\, x}{\gamma(\gamma+1)}\; {}_2F_1\left[\begin{matrix}\alpha+1,\ \beta+1 \\ \gamma+2\end{matrix};\ x\right] = 0. \tag{2·10}$$

Similarly taking the other recurrence formulae for $M_{k,m}$ transform and using the same method, we get the following more recurrence formulae for Gause's hypergeometric function as :

$$(i)\quad \beta x\, {}_2F_1\left[\begin{matrix}\alpha+1,\ \beta+1 \\ \gamma+1\end{matrix};\ x\right] - \gamma\, {}_2F_1\left[\begin{matrix}\alpha+1,\ \beta \\ \gamma\end{matrix};\ x\right] + \gamma\, {}_2F_1\left[\begin{matrix}\alpha,\beta \\ \gamma\end{matrix};\ x\right] = 0$$

$$(ii)\quad \gamma\, {}_2F_1\left[\begin{matrix}\alpha,\beta \\ \gamma\end{matrix};\ x\right] \quad \alpha\, {}_2F_1\left[\begin{matrix}\alpha+1,\beta \\ \gamma+1\end{matrix};\ x\right] - \frac{(\gamma-\alpha)}{(\gamma-1)}\; {}_2F_1\left[\begin{matrix}\alpha,\beta \\ \gamma+1\end{matrix};\ x\right] = 0$$

$$(iii)\quad {}_2F_1\left[\begin{matrix}\alpha+1,\ \beta+1 \\ \gamma+1\end{matrix};\ x\right] \quad \frac{1}{\beta x}\; {}_2F_1\left[\begin{matrix}\alpha,\beta \\ \gamma\end{matrix};\ x\right] - \frac{(\gamma\ \ \alpha)}{\gamma(\gamma+1)}\; {}_2F_1\left[\begin{matrix}\alpha+1,\ \beta+1 \\ \gamma+2\end{matrix};\ x\right] = 0$$

$$(iv)\quad \gamma\, {}_2F_1\left[\begin{matrix}\alpha+1,\beta \\ \gamma+1\end{matrix};\ x\right] - \beta x\, {}_2F_1\left[\begin{matrix}\alpha+1,\beta+1 \\ \gamma+1\end{matrix};\ x\right] - \gamma\, {}_2F_1\left[\begin{matrix}\alpha,\beta \\ \gamma\end{matrix};\ x\right] + \frac{\beta x(\alpha+1)}{(\gamma+1)}$$

$$\times\, {}_2F_1\left[\begin{matrix}\alpha+2,\ \beta+1 \\ \gamma+2\end{matrix};\ x\right] = 0$$

$$(v)\quad \gamma\, {}_2F_1\left[\begin{matrix}\alpha,\beta \\ \gamma+1\end{matrix};\ x\right] + \beta x\, {}_2F_1\left[\begin{matrix}\alpha,\beta+1 \\ \gamma+1\end{matrix};\ x\right] - \frac{(\gamma-\alpha)}{(\gamma+1)}\beta x\, {}_2F_1\left[\begin{matrix}\alpha,\beta+1 \\ \gamma+2\end{matrix};\ x\right]$$

$$+\ \gamma\, {}_2F_1\left[\begin{matrix}\alpha,\beta \\ \gamma\end{matrix};\ x\right] = 0,$$

and

$$(vi)\qquad (\gamma-2\alpha)\, {}_2F_1\left[\begin{matrix}\alpha,\beta \\ \gamma\end{matrix};\ x\right] - \beta\, x\, {}_2F_1\left[\begin{matrix}\alpha,\beta+1 \\ \gamma\end{matrix};\ x\right]$$

$$+\ \beta(\gamma\ \ \alpha)\, {}_2F_1\left[\begin{matrix}\alpha-1,\beta \\ \gamma\end{matrix};\ x\right] + a\, {}_2F_1\left[\begin{matrix}\alpha+1,\beta \\ \gamma\end{matrix};\ x\right] = 0.$$

If in the relation (2·2), (2·3), (2·4) and (2·5), we put $k = m + n + \frac{1}{2}$, L. H. S. reduce to $L_n^{(\alpha)}$ transform, whereas the R. H. S. remains $M_{k,m}$ transform. Thus, it is interesting to note that the sum of two $M_{k,m}$ transforms is equivalent to $L_n^{(\alpha)}$ transform for any function $x^{-\lambda} h(x)$, for which the integrals and the series are convergent.

Further, if we take $k = m + n + \frac{1}{2}$, in (2·6), (2·7) and (2·8), we note that the sum of two $M_{k,m}$ transforms is equivalent to sum of two $L_n^{(\alpha)}$ transforms.

**3. Theorem 2.** If the $M_{k,m}$ transform of $x^{-\lambda} h(x)$ with respect to the first derivative of Whittaker function, $M_{k,m}(Z)$, be $\phi'_{k,m,\lambda}(p)$, that is

$$\phi'_{k,m,\lambda}(p) \overset{M'}{\underset{k,m}{=}} x^{-\lambda}\ h(x)$$

or

$$\phi'_{k,m,\lambda}(p) = p\int_0^\infty (2px)^{-\frac{1}{2}} \frac{d}{dx}\Big[ M_{k,m}(2px)\Big] x^{-\lambda}\ h(x)\ dx, \tag{3·1}$$

then

$$2p\ \phi'_{k,m,\lambda}(p) = k.\ \phi_{k,m,\ \lambda+1}(p) - p\ \phi_{k,m,\lambda}(p) + (\tfrac{1}{2} - k + m)\ \phi_{k-1,m,\lambda+1}(p), \tag{3·2}$$

provided that $R(\rho - \lambda + m + 5/4) > 0$ where $h(x) = 0\ (x^{\rho})$ for small $x$ and $R(p) \geqq Z(p_0) > 0$.

**Proof.** Using the result [2, p. 24]

$z\,M'_{k,m}(z) = (k - z/2)\ M_{k,m}(z) + (\frac{1}{2} - k + m)\ M_{k-1,m}(z)$, in the result (3·1), we obtain the theorem.

**Example.** If we take $h(x) = x^{\sigma} \exp(-\alpha\ x)$, then using the result (2·9) in the result (3·1), we get

$$\phi'_{k,m,\lambda}(p) = \frac{p(2p)^{m-3/4}\,\Gamma(\sigma-\lambda+m+1/4)}{(\alpha+p)^{\sigma-\lambda+m+1/4}} k\ {}_2F_1\left[\begin{matrix}\sigma-\lambda+m+1/4,\ \frac{1}{2}-k+m\\ 2m+1\end{matrix};\ \frac{2p}{\alpha+p}\right]$$

$$-\frac{p(\sigma-\lambda+m+1/4)}{(\alpha+p)}\ {}_2F_1\left[\begin{matrix}\sigma-\lambda+m+5/4,\frac{1}{2}-k+m\\ 2m+1\end{matrix};\ \frac{2p}{\alpha+p}\right] + (\tfrac{1}{2}-k+m)$$

$$\times {}_2F_1\left[\begin{matrix}\sigma-\lambda+m+1/4,\frac{3}{2}-k+m\\ 2m+1\end{matrix};\ \frac{2p}{\alpha+p}\right], \tag{3·3}$$

where $R(\sigma - \lambda + m + 5/4) > 0$ and $(\alpha) > R\ (p) > 0$.

**4. Theorem 3.** If

$$\phi_r(p) \overset{M}{\underset{k+r,m}{=}} h(x)$$

then

$$\sum_{r=0}^{\infty} \frac{1}{\underline{|r}}\left( (1 - \frac{1}{\alpha}\right)^r (\tfrac{1}{2} - k + m)_r\ \phi_r(p)$$

$$= p\ \alpha^k \int_0^\infty (2px)^{-\frac{1}{2}}\ e^{px(1-\alpha)}\ M_{k,m}(2p\alpha x) h(x)\ dx, \tag{4·1}$$

provided that $R(\rho+m+5/4) > 0$, where $h(x) = 0\ (x^{\rho})$ for small $x$, $\alpha$ is positive, $x^{-1/4}\ M_{k,m}(2p_0 x)\ h(x)$ is bounded for $x \geqq 0$ and the series on L. H. S. is uniformly convergent.

**Proof.** Since we have

$$\phi_r(p) = p\int_0^{\infty} (2px)^{-1/4}\ M_{k+r,m}\ (2px)\ h(x)\ dx.$$

Multiplying both sides by $\frac{1}{\underline{|r}}\left(1-\frac{1}{\alpha}\right)^r (\frac{1}{2}-k+m)_r$, taking the sum from zero to infinity and using the result [2, p. 30], we get

$$\alpha^{-k}\ x^{(y-1)x/2} \sum_{r=0}^{\infty} \frac{1}{\underline{|r}}\left(1-\frac{1}{\alpha}\right)^r (\tfrac{1}{2}-k+m)_r\ M_{k+r,m}\ (x) = M_{k,m}\ (\alpha x),\ \alpha \text{ is positive},$$

we obtain the theorem.

The change of the order of summation and integration is justified due to the uniform and absolute convergence of the series and the integral.

**Example.** If we take $h(x) = x^{\sigma}\exp(-\alpha x)$, using the result (2·9) in the result (4·1), we get

$$\int_0^{\infty} x^{\sigma-1/4}\ M_{k,m}\ (2p\alpha x)\ e^{px(1-\alpha)-qx}\ dx = \frac{(2p)^{m+\frac{1}{2}}\ \Gamma(\sigma+m+5/4)}{(\alpha+p)^{\sigma+m+5/4}}\ \alpha^k$$

$$\sum_{r=0}^{\infty} \frac{1}{\underline{|r}}\left(1-\frac{1}{\alpha}\right)^r (\tfrac{1}{2}-k+m)_r\ {}_2F_1\left[\begin{matrix}\sigma+m+5/4,\ \frac{1}{2}-k+m-r\\ 2m+1\end{matrix};\ \frac{2p}{\alpha+p}\right], \tag{4·2}$$

where $R(\sigma+m+5/4) > 0$ and $R(\alpha) > R(p) > 0$.

## 5. Theerem 4. If

$$\phi_r(p)\ \underset{k+r,m}{\overset{M}{=\!=}}\ h(x)$$

then

$$\sum_{r=0}^{\infty} \frac{1}{\underline{|r}}\alpha^r\ (\tfrac{1}{2}+k+m)_r\ \phi_r\ (p) = p\ e^{-\alpha/2}$$

$$\times \int_0^{\infty} (2px)^{-\frac{1}{4}} \left(\frac{x}{x+\alpha}\right)^{-k} M_{k,m}(2px+\alpha)\ h(x)\ dx, \tag{5·1}$$

provided that $R(\rho+m+5/4) > 0$ where $h(x) = 0\ (x^{\rho})$ for small $x$, $\alpha$ is positive, $x^{-\frac{1}{4}}\ M_{k,m}(2p_0 x)\ h(x)$ is bounded for $x \geqq 0$ and the series on L. H. S. is uniformly convergent.

**Proof.** We have

$$\phi_r(p) = p\int_0^{\infty} (2px)^{-1/4}\ M_{k+r,m}(2px)\ h(x)\ dx$$

Multiplying both sides by $\frac{\alpha^r}{L^r}$ $(\frac{1}{2}+k+m)_r$, taking the sum from zero to infinity and using the result [2., p. 29]

$$\sum_{r=0}^{\infty} \frac{\alpha^r}{L^r} \; (\tfrac{1}{2}+k+m)_r \, M_{k+r,m}(2px) = e^{-\alpha/2} \left(\frac{x}{x+\alpha}\right)^{-k} M_{k,m} \,(x+\alpha),$$

where $\alpha$ is positive,we obtain the theorem.

The change of the order of summation and integration is justified due to the uniform and absolute convergence of the series and the integral.

**Example.** If we take $h(x) = x^{\sigma} \exp(\ \ \alpha x)$, and using the result (2·9) in the result (5·1), we get

$$\int_0^{\infty} x^{\sigma-k-1/4}(x+\alpha)^k e^{-\alpha x} M_{k,m}(2px+\alpha)\,dx = \frac{(2p)^{m+\frac{1}{2}}\,\Gamma(\sigma+m+5/4)e^{\alpha/2}}{(\alpha+p)^{\sigma+m+5/4}}$$

$$\sum_{r=0}^{\infty} \frac{\alpha^r}{L^r} \; (\tfrac{1}{2}+k+m)_r \; {}_2F_1\left[\begin{matrix}\sigma+m+5/4,\ \frac{1}{2}-k+m \quad r \\ 2m+1\end{matrix} ;\ \frac{2p}{\alpha+p}\right],$$

where $\qquad R(\sigma + m + 5/4) > 0$ and $R(\alpha) > R(p) > 0.$

**Acknowledgement**

I am highly thankful to Dr. K. C. Sharma, Reader, Department of Mathematics, University of Rajasthan, Jaipur, for his constant guidance and keen interest during the prepration of this paper.

**References**

1. Kulshreshtha, S. K. On an Integral Transform. *Proc. Nat. Acad. Sc., India*, Sec. A, Vol. XXXVI : (Part I), pp. 203-208, (1966).
2. Slater, L. J Confluent Hypergeometric Functions. *Cambridge University Press*, (1960).
3. Varma, R. S. *Current Sciences*, 15 : 16-17, (1947).

# On the Steady Motion of Reiner—Philippoff and Ellis Fluids in an Annulus under Toroidal Pressure Gradient

*By*

S. C. RAJVANSHI

*Department of Mathematics, Malaviya Regional Engineering College, Jaipur*

[Received on 23rd March, 1966]

**Abstract**

In this note the steady motion of Reiner-Philippoff fluid in an annulus under toroidal pressure gradient has been discussed. It has been observed that with increase of toroidal pressure gradient, zero shear stress surface moves towards the inner cylinder, and, with the increase of adverse toroidal pressure gradient, zero-shear-stress surface moves towards the outer cylinder.

The angular velocity profile for Ellis fluid has also been obtained and for small values of the fluid parameter $\overline{\phi}$, it is observed, that the zero-shear stress surface moves towards the inner cylinder with increase in $\alpha$.

## 1. Introduction.

Steady and unsteady tangential flow between two coaxial cylinders may be obtained by application of transverse pressure gradient. The steady case of such a flow has been discussed by Goldstein[1]. The unsteady flows of viscous incompressible fluid obtained by rotating one or both the cylinders with uniform angular velocities have been studied by Bird and Curtiss[2]. Similar problems were also studied by Ghildyal[3], Kapur and Srivastava[4]. The present paper is devoted to the discussion of the steady flow of isotropic, incompressible, non-Newtonian fluid under toroidal pressure gradient through an annulus. The flow equations of non-Newtonian fluids are widely applicable in the fields of chemical and mechanical engineering. We have studied Reiner-Philippoff and Ellis models. The flow behaviour of these models helps in explaining the various properties of a number of industrially important fluids.

Let $\tau_{ij}$ and $e_{ij}$ be deviatoric stress components and strain rate components respectively. The rheological equations for a Reiner-Philippoff fluid is given by[5]

$$\tau_{ij} = \left[\mu_0 + \frac{\mu_\infty - \mu_0}{1 + \dfrac{1}{2\,\tau_0^2} \sum\limits_{l=1}^{3} \sum\limits_{m=1}^{3} \tau_{lm}\, \tau_{ml}}\right] e_{ij}, \qquad (1\cdot1)$$

where $\mu_0$, $\mu_\infty$ and $\tau_0$ are the fluid parameters. The values of these parameters for a number of fluids have been tabulated in[5]. This fluid possesses an interesting property of behaving like a Newtonian fluid when $\tau_0 \to 0$, or $\tau_0 \to \infty$. But for intermediate values of $\tau_0$. the behaviour is definitely non-Newtonian.

The rheological equation for an Ellis fluid is given by[5]

$$e_{ij} = \phi_0\, \tau_{ij} + \phi_1 \left| \left( \sum_{l=1}^{3} \sum_{m=1}^{3} \tau_{lm}\, \tau_{ml} \right)^{\frac{1}{2}} \right|^{\alpha-1} \tau_{ij}, \qquad (1\cdot2)$$

where $\phi_0$, $\phi_1$ and $\alpha$ are the fluid parameters. These parameters for some Ellis fluid are tabulated in[6]. Kapur and Gupta[7] have shown the following interesting results :

(1) when $\phi_1 = 0$, it behaves as a Newtonian fluid,

(2) when $\phi_0 = 0$, it behaves as a power-law fluid,

(3) when $\alpha > 1$, and stress components are small, it approximates to a Newtonian fluid,

(4) when $\alpha < 1$, and stress components are large, it again approximates to the Newtonian fluid,

(5) when $\alpha = 1$, this again represents Newtonian fluid.

## 2. Equations of Motion.

The fluid is contained between two infinitely long coaxial cylinders of radii $a$ and $b$ $(a>b)$. The two cylinders are assumed to be in a steady state of rotation. Hence the velocity components in cylindrical polar coordinates are,

$$u = 0, \qquad v = f(r), \qquad w = 0. \tag{2·1}$$

Therefore, the equations of motion reduce to,

$$\left.\begin{aligned} 0 &= -\frac{\partial p}{\partial r}, \\ 0 &= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, \tau_{r\theta}), \\ 0 &= -\frac{\partial p}{\partial z}. \end{aligned}\right\} \tag{2·2}$$

## 3. Reiner-Philippoff fluid.

The non-zero deviatoric stress component $\tau_{r\theta}$ is given by,

$$\tau_{r\theta} = \left[\mu_0 + \frac{\mu_\infty - \mu_0}{1 + \dfrac{\tau^2_{r\theta}}{\tau_0^{\,2}}}\right] e_{r\theta}. \tag{3·1}$$

The non-dimensional quantities are defined by;

$$\left.\begin{aligned} &\eta = \frac{r}{b}, \quad k = \frac{a}{b}, \quad P = \frac{pb^2\rho}{\mu_0^{\,2}}, \quad T_0 = \frac{\tau_0 b^2\rho}{\mu_0^{\,2}}, \\ &T_{r\theta} = \frac{\tau_{r\theta}\, b^2\rho}{\mu_0^{\,2}}, \quad \epsilon_{r\theta} = \frac{e_{r\theta}\, b^2\rho}{\mu_0}, \quad V = \frac{v\, b\, \rho}{\mu_0}, \\ &\bar{\mu}_\infty = \frac{\mu_\infty}{\mu_0}, \end{aligned}\right\} \tag{3·2}$$

where $\rho$ is the density of the fluid.

From the equations (2·2), (3·1) and (3·2), we have,

$$\left.\begin{aligned}\frac{\partial P}{\partial \theta} &= \frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta^2 T_{r\theta}),\\ \text{and}\qquad T_{r\theta} &= \left(1+\frac{\bar{\mu}_\infty T_0^2 - T_0^2}{T_0^2+T_{r\theta}^2}\right)\varepsilon_{r\theta},\end{aligned}\right\} \tag{3·3}$$

where $$\varepsilon_{r\theta} = \eta\frac{\partial}{\partial \eta}\left(\frac{V}{\eta}\right). \tag{3·4}$$

The uniform toroidal pressure gradient is given by,

$$-\frac{\partial P}{\partial \theta} = A. \tag{3·5}$$

First equation of (3·3) together with (3·5) gives,

$$T_{r\theta} = \frac{A}{2}\left(\frac{\eta_0^2}{\eta^2}-1\right), \tag{3·6}$$

where $\eta = \eta_0$ gives the zero shear stress surface. From the second equation of (3·3), (3·4), and (3·6) we have,

$$\frac{\partial}{\partial \eta}\left(\frac{V}{\eta}\right) = \frac{A(\eta_0^2-\eta^2)}{2\eta^3}\cdot\frac{A^2(\eta_0^2-\eta^2)^2+4T_0^2\eta^4}{A^2(\eta_0^2-\eta^2)^2+4\bar{\mu}_\infty T_0^2\eta^4}. \tag{3·7}$$

We assume $\omega = V/\eta$, where $\omega$ denotes the angular velocity. Let $\omega_1$ and $\omega_2$ be the dimensionless angular velocities of the inner and outer cylinders respectively. Integrating (3·7) we have,

$$\frac{4}{A}(\omega-\omega_1) = \eta_0^2\left(1-\frac{1}{\eta^2}\right) - \frac{1}{1+\bar{\mu}_\infty T'^2_0}\Bigg[(1+T'^2_0)\log\eta^2$$
$$+\frac{T'^2_0}{2}(1-\bar{\mu}_\infty)\log\frac{(\eta_0^2-\eta^2)^2+\bar{\mu}_\infty T'^2_0\eta^4}{\{(\eta_0^2-1)^2+\bar{\mu}_\infty T'^2_0\}\eta^4}$$
$$-\sqrt{\bar{\mu}_\infty T'^6_0}\,(1-\bar{\mu}_\infty)\tan^{-1}\frac{\eta_0^2(\eta^2-1)\sqrt{\bar{\mu}_\infty T'^2_0}}{\eta^2\bar{\mu}_\infty T'^2_0+(\eta_0^2-\eta^2)(\eta_0^2-1)}\Bigg],$$

for $$1 \leqslant \eta \leqslant \eta_0,$$

and,

$$\frac{4}{A}(\omega_2-\omega) = \eta_0^2\left(\frac{1}{\eta^2}-\frac{1}{k^2}\right) - \frac{1}{1+\bar{\mu}_\infty T'^2_0}\Bigg[(1+T'^2_0)\log\frac{k^2}{\eta^2}$$
$$+\frac{T'^2_0}{2}(1-\bar{\mu}_\infty)\log\frac{\{(k^2-\eta_0^2)^2+k^4\bar{\mu}_\infty T'^2_0\}\eta^4}{\{(\eta^2-\eta_0^2)^2+\eta^4\bar{\mu}_\infty T'^2_0\}k^4}$$
$$-\sqrt{\bar{\mu}_\infty T'^6_0}(1-\bar{\mu}_\infty)\tan^{-1}\frac{\eta_0^2(k^2-\eta^2)\sqrt{\bar{\mu}_\infty T'^2_0}}{k^2\eta^2\bar{\mu}_\infty T'^2_0+(k^2-\eta_0^2)(\eta^2-\eta_0^2)}\Bigg], \tag{3·8}$$

for $\qquad \eta_0 \leqslant \eta \leqslant k,$

where $\qquad T_0' = 2\, T_0/A$

At $\eta = \eta_0$, equating the values of $\omega$ obtained from first and second equation of (3·8), we have,

$$\frac{4}{A}(\omega_2-\omega_1) = \eta_1{}^2 \left(1 - \frac{1}{k^2}\right) - \frac{1}{1+\bar{\mu}_\infty\ T'_0{}^2}\Bigg[ (1 + T'_0{}^2) \log k^2 + \frac{T'_0{}^2}{2} (1 - \bar{\mu}_\infty) \log \frac{(k^2-\eta_0{}^2)^2 + k^4\bar{\mu}_\infty\ T'_0{}^2}{k^4\{(\eta_0{}^2-1)^2+\bar{\mu}_\infty\ T'_0{}^2\}} - \sqrt{\bar{\mu}_\infty\ T_0{}'^6}(1 - \bar{\mu}_\infty) \tan^{-1} \frac{\eta_0{}^2(k^2-1)\sqrt{\bar{\mu}_\infty}\ T'_0{}^2}{k^2\bar{\mu}_\infty\ T'_0{}^2 - (k^2-\eta_0{}^2)(\eta_0{}^2-1)} \Bigg]. \tag{3·9}$$

From equation (3·9), it is clear that, $\eta_0$ cannot be determined explicitly. However, in limiting cases, when $T_0'$ is very small or very large, it is possible to determine $\eta_0$.

When $T_0'$ is small, we assume

$$\eta_0{}^2 = \eta_1{}^2 + T_0{}'^2\, \eta_2{}^2. \tag{3·10}$$

Substituting (3·10) in (3·9) and neglecting fourth and higher powers of $T_0'$ we obtain,

$$\left.\begin{aligned} \eta_1{}^2 &= \frac{4k^2(\omega_2-\omega_1) + Ak^2 \log k^2}{A(k^2-1)}, \\ \text{and} \quad \eta_2{}^2 &= \frac{k^2(1-\bar{\mu}_\infty)}{k^2-1} \log \frac{k^2-\eta_1{}^2}{\eta_1{}^2-1}. \end{aligned}\right\} \tag{3·11}$$

When $T_0'$ is large, we assume

$$\eta_0{}^2 = \eta'_1{}^2 + \frac{\eta'_2{}^2}{T'_0{}^2} \tag{3·12}$$

Substituting (3·12) in (3·9) and neglecting fourth and higher powers of $1/T_0'$ we get,

$$\left.\begin{aligned} \eta'_1{}^2 &= \frac{4\bar{\mu}_\infty\ k^2(\omega_2-\omega_1) + Ak^2 \log k^2}{A(k^2-1)}, \\ \eta'_2{}^2 &= \frac{1-\bar{\mu}_\infty}{6k^4\bar{\mu}_\infty\ (k^2-1)}(18\, k^6\eta'_1{}^2+2k^6\eta'_1{}^6 - 9k^6\eta'_1{}^4 \\ &\quad - 18k^4\eta'_1{}^2 + 9k^2\eta'_1{}^4 - 2\eta'_1{}^6 + \log k^2). \end{aligned}\right\} \tag{3·13}$$

## 4. Numerical Results.

For numerical computation, we assume, the cylinders to be rotating in opposite directions and let us take,

$$\omega_1 = -\,0{\cdot}4,\ \omega_2 = 0{\cdot}1,\ k = 2. \tag{4·1}$$

The fluid with small value of $\tau_0$ is Molten sulphur and its parameters are[5],

$$\mu_0 = 0{\cdot}215,\ \mu_\infty = 0{\cdot}0105 \text{ and } \tau_0 = 0{\cdot}073. \tag{4·2}$$

The fluid with large value of $\tau_0$ is 0·4% Polysterene in Tetralin and its parameters are[5]

$$\mu_0 = 4{\cdot}0, \; \mu_\infty = 1{\cdot}0 \text{ and } \tau_0 = 500. \tag{4·3}$$

For Molten Sulphur, $\eta_0$ is obtained from equations (3·10) and (3·11). For 0·4% Polysterene in Tetralin, $\eta_0$ is obtained from (3·12) and (3·13). Angular velocity profiles have been drawn against $\eta$. We observe from the graph that for Molten Sulphur as well as 0·4% Polysterene in Tetralin, with the increase of toroidal pressure gradient *i.e.* for positive values of $A$, the zero shear-surface moves towards the inner cylinder. With the numerical increase of adverse toroidal pressure gradient *i.e.* for negative values of $A$, the zero shear-surface moves towards the outer cylinder. The variations in angular velocity are more marked in fluid with large $\tau_0$ as we move from inner cylinder to outer cylinder.

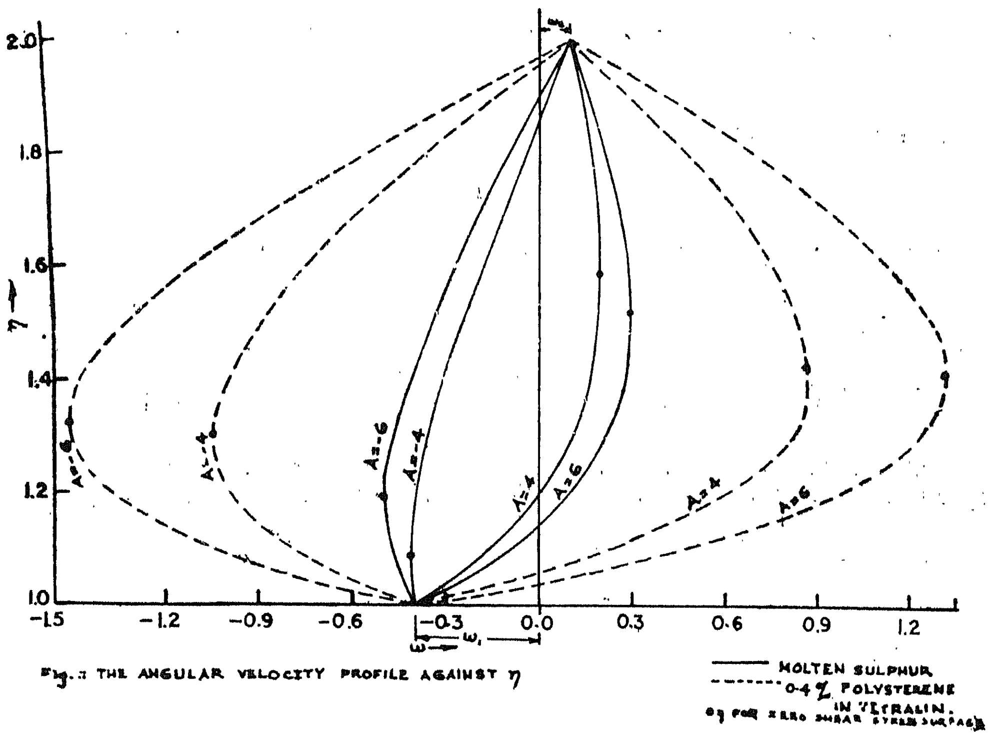



Fig. 3 THE ANGULAR VELOCITY PROFILE AGAINST $\eta$

## 5. Ellis Fluid.

The non-zero deviatoric stress component $\tau_{r\theta}$ is given by.

$$e_{r\theta} = \phi_0 \, \tau_{r\theta} + \phi_1 \, 2^{(\alpha-1)/2} \left| \tau_{r\theta} \right|^{\alpha-1} \tau_{r\theta}. \tag{5·1}$$

Introducing the following non-dimensional quantities :

$$\left.\begin{aligned} \eta = \frac{r}{b},\ \bar{\tau}_{r\theta} = \tau_{r\theta}\,\rho b^2\,\phi_0{}^2,\ \bar{e}_{r\theta} = e_{r\theta}\,\rho b^2\phi_0, \\ \bar{P} = p\,\rho b^2\,\phi_0{}^2,\ \bar{\phi} = \frac{\phi_1\,.\,2^{(\alpha-1)/2}}{\phi_0{}^{2\alpha-1}(\rho b^2)^{\alpha-1}},\ V = v\rho b\,\phi_0. \end{aligned}\right\} \tag{5·2}$$

From equations (2·2), (5·1) and (5·2) we have,

$$\frac{\partial \bar{P}}{\partial \theta} = \frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta^2\,\bar{\tau}_{r\theta}), \tag{5·3}$$

and

$$\bar{e}_{r\theta} = \bar{\tau}_{r\theta} + \bar{\phi}\,|\,\bar{\tau}_{r\theta}\,|^{\alpha-1}\,\bar{\tau}_{r\theta}.$$

Proceeding as in section 3, we assume, the toroidal pressure gradient

$$-\frac{\partial \bar{P}}{\partial \theta} = \bar{A},$$

and from (5·3), we have,

$$\bar{\tau}_{r\theta} = \frac{\bar{A}}{2}\left(\frac{\eta_0{}^2}{\eta^2} - 1\right). \tag{5·4}$$

From equation (5·4) and second equation of (5·3), we have,

$$\frac{\partial \omega}{\partial \eta} = \frac{\bar{A}}{2\eta}\left(\frac{\eta_0{}^2}{\eta^2} - 1\right) + \bar{\phi}\left(\frac{\bar{A}}{2}\right)^{\alpha}\frac{1}{\eta}\left(\frac{\eta_0{}^2}{\eta^2} - 1\right)^{\alpha},\ \text{for } 1 \leqslant \eta \leqslant \eta_0,$$

and

$$\frac{\partial \omega}{\partial \eta} = -\frac{\bar{A}}{2\eta}\left(1 - \frac{\eta_0{}^2}{\eta^2}\right) - \bar{\phi}\left(\frac{\bar{A}}{2}\right)^{\alpha}\frac{1}{\eta}\left(1 - \frac{\eta_0{}^2}{\eta^2}\right)^{\alpha},\ \text{for } \eta_0 \leqslant \eta \leqslant k. \tag{5·5}$$

Let $\omega_1$ and $\omega_2$ be the angular velocities of the inner and outer cylinders respectively. Integrating equations (5·5) for fractional values of $\alpha$ we have,

$$\omega - \omega_1 = \frac{\bar{A}}{2}\left[\frac{\eta_0{}^2}{2\eta^2}(\eta^2 - 1) - \log\eta\right]$$

$$+ \tfrac{1}{2}\,\bar{\phi}\left(\frac{\bar{A}}{2}\right)^{\alpha}\left[B(1/\eta_0{}^2)(-\alpha,\ 1+\alpha) - B(\eta^2/\eta_0{}^2)(-\alpha,\ 1+\alpha)\right],$$

for $$1 \leqslant \eta \leqslant \eta_0,$$

and

$$\omega_2 - \omega = \frac{\bar{A}}{2}\left[\frac{\eta_0{}^2(k^2 - \eta^2)}{2\eta^2 k^2} - \log\frac{k}{\eta}\right]$$

$$+ \tfrac{1}{2}\,\bar{\phi}\left(\frac{\bar{A}}{2}\right)^{\alpha}\left[B(\eta_0{}^2/\eta^2)(0,\ 1+\alpha) - B(\eta_0{}^2/k^2)(0,\ 1+\alpha)\right], \tag{5·6}$$

for $$\eta_0 \leqslant \eta \leqslant k.$$

Equating the values of $\omega$ given by two equations of (5·6) at $\eta = \eta_0$, we obtain,

$$\omega_2 - \omega_1 = \frac{\bar{A}}{4}\left[\frac{\eta_0^2}{k^2}(k^2 - 1) - \log k^2\right]$$

$$+\tfrac{1}{2}\,\bar{\phi}\left(\frac{\bar{A}}{2}\right)^{\alpha}\left[B\,(1/\eta_0^2)\,(\alpha, 1+\alpha) - B\eta_0^2/k^2)(0, 1+\alpha)\right], \qquad (5·7)$$

where
$$B_x(m, n) = \int_x^1 t^{m-1}(1-t)^{n-1}\,dt. \qquad (5·8)$$

The zero shear stress surface is determined by (5·7). Integrating equations 5·5) for integral values of $\alpha$, we have,

$$\omega - \omega_1 = \frac{\bar{A}}{2}\left[\frac{\eta_0^2}{2\eta^2}\,\eta^2 - 1) - \log\eta\right] + \bar{\phi}\left(\frac{\bar{A}}{2}\right)^{\alpha}\left[(-1)^{\alpha}\log\eta\right.$$

$$\left. + \sum_{m=0}^{\alpha-1}(-1)^m\,{}^{\alpha}c_m\,\eta_0^{2\alpha-2m}\left(\frac{\eta^{2m-2\alpha}-1}{2m-2\alpha}\right)\right],$$

for
$$1 \leqslant \eta \leqslant \eta_0,$$

$$\omega_2 - \omega = \frac{\bar{A}}{2}\left[\frac{\eta_0^2\,(k^2-\eta^2)}{2\eta^2 k^2} - \log\frac{k}{\eta}\right] - \bar{\phi}\left(\frac{\bar{A}}{2}\right)^{\alpha}\left[\log\frac{k}{\eta}\right.$$

$$\left. + \sum_{m=1}^{\alpha}\frac{(-1)^{m+1}}{2m}\,{}^{\alpha}c_m\,\eta_0^{2m}\left(\frac{1}{k^{2m}} - \frac{1}{\eta^{2m}}\right)\right], \qquad (5·9)$$

for
$$\eta_0 \leqslant \eta \leqslant k.$$

Equating the values of $\omega$ from both the equations of (5·9) at $\eta = \eta_0$, we get

$$\omega_2 - \omega_1 = \frac{\bar{A}}{2}\left[\frac{\eta_0^2\,(k^2-1)}{2\,k^2} - \log k\right.$$

$$\left. + \bar{\phi}\left(\frac{\bar{A}}{2}\right)^{\alpha}\left[D + \sum_{m=1}^{\alpha-1}(-1)^m\,{}^{\alpha}c_m\left(\frac{1-\eta_0^{2\alpha-2m}}{2m-2\alpha} + \frac{\eta_0^{2m}-k^{2m}}{2m\,k^{2m}}\right)\right]\right], \qquad (5·10)$$

where

$$\left.\begin{aligned} D &= -\log k + \frac{\eta_0^{2\alpha}(k^{2\alpha}-1)}{2\alpha\,k^{2\alpha}}, && \text{when } \alpha \text{ is odd,} \\ &= \log\frac{\eta_0^2}{k} + \frac{k^{2\alpha}\,\eta_0^{2\alpha} - 2k^{2\alpha} + \eta_0^{2\alpha}}{2\alpha\,k^{2\alpha}}, && \text{when } \alpha \text{ is even.} \end{aligned}\right\} \qquad (5·11)$$

Equations (5·10) and (5·11) determine $\eta_0$.

The effect of Ellis fluid parameter $\alpha$ on the zero shear stress surface is determined by considering the perturbation in the other parameter $\bar{\phi}$ of the fluid.

Let
$$\eta_0 = \eta_1 + \bar{\phi}\,\eta_2, \qquad (5·12)$$
where $\bar{\phi}$ is small.

For numerical calculation we assume

$$\omega_1 = -0{\cdot}4, \quad \omega_2 = 0{\cdot}1, \quad k = 2{\cdot}0, \text{ and } \bar{A} = 4{\cdot}0. \tag{5·13}$$

From equations (5·10) to (5·13) the values of $\eta_1$ and $\eta_2$ are tabulated below for the various values of $a$.

| $a$ | $\eta_1$ | $\eta_2$ |
|---|---|---|
| 1 | 1·47707 | − 0·05642 |
| 2 | 1·47707 | − 0·11209 |
| 3 | 1·47707 | − 0·21347 |
| 4 | 1·47707 | − 0·40760 |

We infer from the above table that for Ellis fluid with small $\phi$, as $a$ increases, the zero shearstress surface moves towards the inner cylinder.

**Acknowledgements.**

The author is greatly indebted to Dr. P. D. Verma for suggesting the problem and for constant help and guidance throughout the preparation of this paper. The author is also thankful to Professor P. L. Bhatnagar for his encouragement during this investigation.

**References**

1. Goldstein, S. Modern development in fluid dynamics, Vol. I. *Clarendon Press, Oxford*, 315, (1938).
2. Bird, R. B. and Curtiss, C. F. *Chem. Engg. Science*, 2 : 108–113, (1959).
3. Ghildyal, C. D. *ZAMM*, 39 : 473-476, (1959).
4. Kapur, J. N. and Srivastava, P. N. *Ganita*, 13 : 17–24, (1962).
5. Bird, R. B., Stewart, W. E. and Lightfoot, E. N. Transport Phenomena, *John Wiley and Sons, New York*, 14, (1961).
6. Slattery, J. C. *Doctoral Thesis, University of Wisconsis*, (1959)
7. Kapur, J. N. and Gupta, R. C. *Archiwum Mechaniki Stosowanej*, 17 : 31–42, (1965).

# $M_{k,m}$ Transform of Two Variab'es

*By*

S. K. KULSHRESHTHA*

*Department of Mathematics, University of Rajasthan, Jaipur*

[Received on 20th April, 1966]

**Abstract**

In this paper $M_{k,m}$ transform of two variables is defined. The theorems obtained by S.K. Bose[2] and C. B. Rathie[6] for Laplace transform of two variables are generalised. Some of the theorems obtained here are analogous to other transforms. Images of some functions in $M_{k,m}$ transform and Laplace transform of two variables are also obtained and in the last differential and integral properties of the said transform are discussed.

## 1. Introduction

I have generalised the Laplace transform defined by the integral equation

$$\psi(p) = p \int_0^{\infty} e^{-pt}\, h(t)\, dt \tag{1·1}$$

in the form[4]

$$\phi(p) = p \int_0^{\infty} (2pt)^{-1/4}\, M_{k,m}(2pt)\, h(t)\, dt \tag{1·2}$$

When $k = -m = 1/4$, (1·2) reduces to (1·1) by virtue of the well known identity

$$M_{1/4,-1/4}(x) \equiv x^{\frac{1}{4}} e^{-x/2} \tag{1·3}$$

Now, in this paper, we define $M_{k,m}$ transform of two variables as

$$\phi(p, q) = p\, q \int_0^{\infty} \int_0^{\infty} (2px)^{-1/4} (2qy)^{-1/4} M_{k,m}(2px)\, M_{k',m'}(2qy)\, h(x,y)\, dx\, dy,$$

$$R(p,q) > 0, \tag{1·4}$$

which is the generalisation of the Laplace transform of two variables defined as

$$\phi(p,q) = pq \int_0^{\infty} \int_0^{\infty} e^{-(px+qy)}\, h(x,y)\, dx\, dy. \tag{1·5}$$

Integral equation (1·4) reduces to (1·5) when $k = k' = -m = -m' = 1/4$.

We shall denote (1·4) and (1·2) symbolically as

$$\phi(p,q) \,\underset{k',m'}{\overset{k,m}{=\!=}}\, h(x,y) \text{ and } \phi(p) \,\underset{k,m}{\overset{M}{=\!=}}\, h(t).$$

respectively and (1·5) and (1·1) as usual shall be denoted as

$$\phi(p,q) \;\ddot{=}\; h(x,y) \qquad \text{and} \qquad \psi(p) \doteqdot h(t).$$

**Present address* : Department of Mathematics, Samrat Ashok Technological Institute, Vidisha (M. P.) India.

The object of this paper is to study the rules, obtain the images of some of the functions in $M_{k,m}$ and Laplace transform and to give some properties of this generalisation in two variables. Also, some differential and integral properties of the said transform are obtained in this note. Some theorems on this generalised transform are also obtained. Particular cases of these theorems give rise to the theorems obtained by Bose[1], Rathie[6] etc.

**2. Theorem 1.** If

$$\phi_1(p,q) \, \frac{k,m}{\overline{\overline{k',m'}}} \; h_1(x,y)$$

and

$$\phi_2(p,q) \, \frac{k,m}{\overline{\overline{k',m'}}} \; h_2(x,y)$$

then

$$\int_0^\infty \int_0^\infty \phi_1(u,v)\, h_2(u,v) \frac{du}{u} \frac{dv}{v} = \int_0^\infty \int_0^\infty \phi_2(u,v)\, h_1(u,v) \frac{du}{u} \frac{du}{v}, \tag{2·1}$$

provided that the order of integration may be inverted (2s1)

**Proof.**

Here, we have

$$\phi_1(u,v) = uv \int_0^\infty \int_0^\infty (2ut)^{-1/4} (2vs)^{-1/4} M_{k,m}(2ut)\, M_{k',m'}(2vs)\, h_1(t,s)\, dt\, ds \tag{2·2}$$

and

$$\phi_2(t,s) = ts \int_0^\infty \int_0^\infty (2tu)^{-1/4} (2sv)^{-1/4} M_{k,m}(2tu)\, M_{k',m'}(2sv)\, h_2(u,v)\, du\, dv \tag{2·3}$$

therefore

$$\int_0^\infty \int_0^\infty \phi_1(u,u)\, h_2(u,u) \frac{du}{u} \frac{dv}{v}$$

$$= \int_0^\infty \int_0^\infty h_2(u,v)\, du\, dv \int_0^\infty \int_0^\infty (2ut)^{-1/4} (2vs)^{-1/4} M_{k,m}(2ut)\, M_{k',m'}(2vs)\, h_1(t,s)\, dt\, ds$$

$$= \int_0^\infty \int_0^\infty h_1(t,s) \left[ \int_0^\infty \int_0^\infty (2ut)^{-1/4} (2vs)^{-1/4} M_{k,m}(2ut)\, M_{k',m'}(2vs)\, h_1(u,v)\, du\, dv \right] ds\, dt$$

$$= \int_0^\infty \int_0^\infty h_1(t,s)\, \phi_2(t,s) \frac{ds}{s} \frac{dt}{t},$$

provided the change of the order of integration is permitted.

Regarding this change, we note that the conditions for the absolute and uniform convergence of (2·2) are $R(\eta_1 + m + 5/4) > 0$, $h(\eta_2 + m' + 5/4) > 0$ where $h_1(t,s) = 0\,(t^{\eta_1})$ for small $t$ and $h_1(t,s) = 0\,(s^{\eta_2})$ for small $s$, an additional condition, if required $h_1(t,s) = 0\,[e^{-t^{\lambda_1}}]$, $R(\lambda_1) > 0$ for large $t$ and $h_1(t,s) = 0\,[e^{-s^{\lambda_2}}]$, $R(\lambda_2) > 0$, for large $s$. Integral (2·3) will be absolutely and uniformly convergent if $R(\eta_3 + m + 5/4) > 0$, $R(\eta_4 + m' + 5/4) > 0$ where $h_2(t,s) = 0\,(t^{\eta_3})$ for

small $t$ and $h_2(t,s) = 0\,(s^{\eta_4})$ for small $s$ and an additional condition if required is $h_2(t,s) = 0\,[e^{-t^{\lambda_3}}]$, $R(\lambda_3) > 0$, for large $t$ and $h_2(t,s) = 0\,[e^{-s^{\lambda_4}}]$, $R(\lambda_4) > 0$, for large $s$. For the justification in the change of the order of integration, we have imposed hard condition but these conditions may be relaxed.

**3. Theorem 2.** If

$$h(x, y) = h_1(x)\, h_2(y)$$

and if

$$\phi_1(p) \quad \frac{M}{k,m} \quad h_1(x),$$

$$\phi_2(q) \quad \frac{M}{k',m'} \quad h_2(y)$$

then

$$h(x,y) \quad \frac{k,m}{k',m'} \quad \phi(p,q) = \phi_1(p).\ \phi_2(q). \tag{3·1}$$

provided that the integrals are absolutely convergent.

**Example 1.**

Taking $h(x,y) = x^{v_1} y^{v_2} e^{-(\alpha x+\beta y)}$ and using [3, p. 215], we find that

$$x^{v_1} y^{v_2} e^{-(\alpha x+\beta y)} \frac{k,m}{k',m'} \frac{pq(2p)^{m+1/4}(2q)^{m'+1/4}\,\Gamma(v_1+m+5/4)\,\Gamma(v_2+m'+5/4)}{(\alpha+p)^{v_1+m+5/4}(\beta+q)^{v_2+m'+5/4}}$$

$${}_2F_1\left[\begin{matrix}\frac{1}{2}-k+m,\ v_1+m+5/4\\ 2m+1\end{matrix};\ \frac{2p}{\alpha+p}\right] {}_2F_1\left[\begin{matrix}\frac{1}{2}-k'+m',\ v_2+m'+5/4\\ 2m'+1\end{matrix};\ \frac{2q}{\beta+q}\right], \quad 3·2)$$

for $R(p,q) > 0$, $R(v_1 + m + 5/4) > 0$ and $R(v_2 + m' + 5/4) > 0$.

In this result, if we take $k = -m + 1/4$, $k' - m' = 1/4$ and $\alpha = \beta = 0$, we get the known result [5, p. 62]

$$x^{v_1} y^{v_2} \doteqdot \frac{\Gamma(v_1+1)\,\Gamma(v_2+1)}{p^{v_1}\, q^{v_2}} \tag{3·3}$$

Again, taking the same $h(x,y)$ and using the result [1, p. 18], we get

$$x^{v_1} y^{v_2} e^{-(\alpha x+\beta y)} \frac{n+m+\frac{1}{2},\, m}{n'+m'+\frac{1}{2},m'} \frac{pq(-1)^{n+n'}\Gamma(2m+1)\Gamma(2m'+1)\Gamma(v_1\pm m+5/4)\Gamma(v_2\pm m'+5/4)}{(2p)^{v_1+1}(2q)^{v_2+1}\Gamma(2m+n+1)\Gamma(2m'+n'+1)\Gamma(v_1-m-n+5/4)}$$

$$\times \frac{1}{\Gamma(v_2-m'-n'+5/4)}\ {}_2F_1\left[\begin{matrix}v_1\pm m+5/4\\ v_1-m-n+5/4\end{matrix};\ \frac{1}{2}-\frac{\alpha}{2p}\right] {}_2F_1\left[\begin{matrix}v_2\pm m'+5/4\\ v_2-m'-n'+5/4\end{matrix};\ \frac{1}{2}-\frac{\beta}{2q}\right] \quad (3·4$$

where $R(v_1 \pm m + 5/4) > 0$, $R(v_2 \pm m' + 5/4) > 0$, $R(p_1 q) > 0$ and $n$ and $n'$ are positive integers.

**Example 2.**

Taking $h(x,y) = x^{\sigma}y^{\sigma 1}(1+x)^{-\frac{1}{2}}(1+y)^{-\frac{1}{2}}$ and using the result [1., p. 17], we get

$$x^{\sigma}y^{\tau'}(1+x)^{-\frac{1}{2}}(1+y)^{-\frac{1}{2}} \ \underset{n'+m'+\frac{1}{2},m'}{\overset{n+m+\frac{1}{2},m}{=\!=}} \ \frac{pq\,\Gamma(n+1)\,\Gamma(2m+1)\,\Gamma(n'+1)\,\Gamma(2m'+1)\,e^{\frac{1}{2}(p+q)}}{p^{\frac{1}{2}(\sigma+m+7/4)}\,q^{\frac{1}{2}(\tau'+m'+7/4)}\,(2p)^{-m-1/4}\,(2q)^{-m'1/4}}$$

$$\sum_{r0}^{n} \frac{(-2\sqrt{p})^{r}\,\Gamma(\sigma+m+5/4+r)}{\underline{|(n-r}\,\underline{|\,r}\;\Gamma(1+2m+r)}\,W_{-\frac{1}{2}(\sigma+m+r+7/4),\,\frac{1}{2}(\sigma+m+r+7/4)}(p)$$

$$\times\sum_{r0}^{n} \frac{(-2\sqrt{q})^{r}\,\Gamma(\sigma'+m'+5/4+r)}{\underline{|(n'-r}\,\underline{|\,r}\;\Gamma(1+2m'+r)}\,W_{-\frac{1}{2}(\sigma'+m'+r+7/4),\,\frac{1}{2}(\sigma'+m'+r+7/4)}(q), \tag{3·5}$$

where $\quad R(\sigma+m+5/4)>0,\ R(\sigma'+m'+5/4)>0$ and $R(p,q)>0$.

In this result, if we take $n = n' = 0$ and $m = m' = -1/4$, we get the result as

$$x^{\sigma}y^{\tau'}(1+x)^{-\frac{1}{2}}(1+y)^{-\frac{1}{2}}$$

$$\doteqdot \frac{\sqrt{2}\,e^{\frac{1}{2}(p+q)}\,\Gamma(\sigma+1)\,\Gamma(\sigma'+1)}{p^{\sigma/2-1/4}\,q^{\sigma'/2-1/4}}$$

$$\times W_{-\frac{1}{2}(\sigma_1+3/2),\,\frac{1}{2}(\sigma_1+3/2)}(p)\;W_{-\frac{1}{2}(\sigma'+3/2),\,\frac{1}{2}(\sigma'+3/2)}(q), \tag{3·6}$$

where $\quad R(\sigma+1)>0,\quad R(\sigma'+1)>0\ \ R(p,q)>0.$

**4. Theorem 3.** If

$$\phi(p,q) \ \underset{k',m'}{\overset{k,m}{=\!=}} \ h(x,y)$$

then

$$\phi(p/\alpha,\,q/\beta) \underset{k',m'}{\overset{k,m}{=\!=}} \ h(\alpha x,\,\beta y), \tag{4·1}$$

provided the integrals involved are absolutely convergent.

**5. Theorem 4.** If

$$\phi_1(p,q) \ \underset{k',m'}{\overset{k,m}{=\!=}} \ h_1(x,y)$$

$$\phi_2(p,q) \ \underset{k',m'}{\overset{k',m}{=\!=}} \ h_2(x,y)$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\phi_n(p,q) \ \underset{k',m'}{\overset{k,m}{=\!=}} \ h_n(x,y)$$

then

$$\sum_{r=1}^{n} \phi_r(p,q) \quad \overset{k,m}{\underset{k',m'}{=}} \quad \sum_{r=1}^{n} h_r(x,y) \tag{5·1}$$

**6. Theorem 5.** If

$$\phi(p,q) \doteqdot x^{v-1} y^{v'-1} \psi(x,y)$$

and

$$\psi(p,q) \quad \overset{k,m}{\underset{k',m'}{=}} \quad g(x,y) \tag{6·1}$$

then

$$\phi(p,q) = \frac{pq\,\Gamma(v+m+5/4)\,\Gamma(v'+m'+5/4)}{2^{-m-m'-\frac{1}{2}}} \int_0^\infty \int_0^\infty \frac{x^{m+1/4}\, y^{m'+1/4}\, g(x,y)}{(p+x)^{v+m+5/4}(q+y)^{v'+m'+5/4}}$$

$$ {}_2F_1 \left[ \begin{matrix} \frac{1}{2}-k+m,\ 5/4+m+v \\ 2m+1 \end{matrix} ;\ \frac{2x}{p+x} \right] {}_2F_1 \left[ \begin{matrix} \frac{1}{2}-k'+m',\ 5/4+m'+v' \\ 2m'+1 \end{matrix} ;\ \frac{2y}{q+y} \right) dx\, dy, \tag{6·2}$$

provided that $R(p,q) > 0$, $R(v + m + 5/4) > 0$, $R(v' + m' + 5/4) > 0$ and the integrals involved in (6·2) are convergent.

**Proof.**

Using the result (3·2) and (6·1) in the theorem 1, we get

$$\int_0^\infty \int_0^\infty x^{v-1} y^{v'-1} e^{-(\alpha x+\beta y)} \psi(x,y)\, dx\, dy = \Gamma(v + m + 5/4)\,\Gamma(v' + m' + 5/4)$$

$$\int_0^\infty \int_0^\infty (2x)^{m+1/4} (2y)^{m'+1/4} (\alpha + x)^{-(v+m+5/4)} (\beta+y)^{-(v'+m'+5/4)} g(x,y)$$

$$\times\ {}_2F_1 \left[ \begin{matrix} \frac{1}{2}-k+m,\ v+m+5/4 \\ 2m+1 \end{matrix} ;\ \frac{2x}{\alpha+x} \right] {}_2F_1 \left[ \begin{matrix} \frac{1}{2}-k'+m',\ v'+m'+5/4 \\ 2m+1 \end{matrix} ;\ \frac{2y}{\beta+y} \right] dx\, dy$$

Multiplying both sides by $\alpha\ \beta$ and finally on replacing $\alpha$ by $p$ and $\beta$ by $q$, we obtain the theorem.

**Corollary.** Taking $k = k' = -m = -m' = 1/4$ in the theorem, we obtain the theorem due to Rathie [6, p. 47]

**7. Theorem 6.** If

$$\phi(p,q) \doteqdot {}_2F_1 \left[ \begin{matrix} x\pm m+5/4 \\ x\cdot m-n+5/4 \end{matrix} ;\ \tfrac{1}{2} \right] {}_2F_1 \left[ \begin{matrix} y\pm m'+5/4 \\ y-m'-n'+5/4 \end{matrix} ;\ \tfrac{1}{2} \right] f(x,y)$$

then

$$\frac{\phi(\log p,\, \log q)}{\log p.\, \log q} \quad \overset{n+m+\frac{1}{2},m}{\underset{n'+m'+\frac{1}{2},m'}{=}} \quad \frac{4\,\Gamma(n+2m+1)\,\Gamma(n'+2m'+1)}{(-1)^{n+n'}\,\Gamma(2m+1)\,\Gamma(2m'+1)}$$

$$\times \int_0^\infty \int_0^\infty \frac{\Gamma(s-m-n+5/4\,\Gamma(t-m'-n'+5/4)}{\Gamma(s\pm m+5/4\,\Gamma(t\pm m'+5/4)} (2x)^s (2y)^t f(s,t)\, ds\, dt \tag{7·1}$$

provided that the integrals involved are absolutely convergent.

**Proof.**

We have

$$\phi(p,q) = pq \int_0^\infty \int_0^\infty e^{-(px+qy)} \, {}_2F_1 \left[ \begin{matrix} x \pm m + 5/4 \\ x-m-n+5/4 \end{matrix} , \tfrac{1}{2} \right] {}_2F_1 \left[ \begin{matrix} y \pm m' + 5/4 \\ y-m'-n'+5/4 \end{matrix} ; \tfrac{1}{2} \right] f(x,y) \, dx \, dy$$

On replacing $p$ and $q$ by $\log p$ and $\log q$ respectively, we get

$$\phi(\log p, \log q) = \log p . \log q \int_0^\infty \int_0^\infty p^{-s} q^{-t} f(s, t)$$

$$\times \; {}_2F_1 \left[ \begin{matrix} s \pm m + 5/4 \\ s-m-n+5/4 \end{matrix} ; \tfrac{1}{2} \right] {}_2F_1 \left[ \begin{matrix} t \pm m' + 5/4 \\ t-m'-n'+5/4 \end{matrix} ; \tfrac{1}{2} \right] ds \, dt \tag{7·2}$$

Now, using the result the result [1, p. 18]

$$p^{-s} \, {}_2F_1 \left[ \begin{matrix} s \pm m + 5/4 \\ s-m-n+5/4 \end{matrix} ; \tfrac{1}{2} \right]$$

$$\underset{n+m+\frac{1}{2},m}{\overset{M}{=\!=}} \quad \frac{2\,(2x)^s \, \Gamma(2m+n+1) \, \Gamma(s-m-n+5/4)}{(-1)^n \, \Gamma(2m+1) \, \Gamma(s \pm m+5/4)} \tag{7·3}$$

in the result (7·2), we obtain the theorem.

**Corollary.**

In the theorem, if we take $n = 0$ and $m = m' = -1/4$, we get the theorem due to Bose [2, p. 175].

## 8. Differential Property.

Differentiating the equality (4·1) with respect to $a$ and $b$ partially and then putting $a$ and $b$ equal to unity, we get

$$-p \frac{\partial}{\partial p} \left[ \phi(p, q) \right] \underset{k',m'}{\overset{k,m}{=\!=}} x \frac{\partial}{\partial x} \left[ h(x,y) \right] \tag{8·1}$$

and

$$-q \frac{\partial}{\partial q} \left[ \phi(p, q) \right] \underset{k',m'}{\overset{k,m}{=\!=}} y \frac{\partial}{\partial y} \left[ h(x,y) \right] \tag{8·2}$$

provided that both sides exist and are continuous.

**9. Integral Property.** After dividing by $a$ and $b$, if we integrate the equality (4·1) with respect to $a$ and $b$ respectively between the limits 0 and 1, we get

$$\int_p^\infty \phi(a,q) \frac{da}{a} \underset{k',m'}{\overset{k,m}{=\!=}} \int_0^x h(a,y) \frac{da}{a}$$

$$\int_q^\infty \phi(p,b) \frac{db}{b} \underset{k',m'}{\overset{k,m}{=\!=}} \int_0^y h(x,b) \frac{db}{b}$$

and

$$\int_p^\infty \int_q^\infty \phi(a,b) \frac{da}{a} \frac{db}{b} \underset{k',m',}{\overset{k,m}{=\!=}} \int_0^x \int_0^y h(a,b) \frac{da}{a} \frac{db}{b}$$

provided that the integrals are convergent.

Instead of taking the limits from 0 to 1, if we take the limits from 0 to $\infty$ ; we get

$$
\left.\begin{aligned}
&\int_0^\infty \phi(a,q)\,\frac{da}{a} \qquad \frac{k,m}{k',m'} \qquad \int_0^\infty h(a,y)\,\frac{da}{a}\\
&\int_0^\infty \phi(p,b)\,\frac{db}{b} \qquad \frac{k,m}{k',m'} \qquad \int_0^\infty h(x,b)\,\frac{db}{b}\\
&\text{and}\\
&\int_0^\infty\int_0^\infty \phi(a,b)\,\frac{da}{a}\,\frac{db}{b} \quad \frac{k,m}{k',m'} \quad \int_0^\infty\int_0^\infty h(a,b)\,\frac{da}{a}\,\frac{db}{b}
\end{aligned}\right\} \qquad (9\cdot2)
$$

provided that the integrals are convergent.

Substracting (9·1) from (9·2), we get

$$
\left.\begin{aligned}
&\int_0^p \phi(a,q)\,\frac{da}{a} \qquad \frac{k,m}{k',m'} \qquad \int_x^\infty h(a,y)\,\frac{da}{a}\\
&\int_0^q \phi(p,b)\,\frac{db}{b} \qquad \frac{k,m}{k',m'} \qquad \int_y^\infty h(x,b)\,\frac{db}{b}\\
&\text{and}\\
&\int_0^p\int_0^q \phi(a,b)\,\frac{da}{a}\,\frac{db}{b} \quad \frac{k,m}{k',m'} \quad \int_x^\infty\int_y^\infty h(a,b)\,\frac{da}{a}\,\frac{db}{b}
\end{aligned}\right\} \qquad (9\cdot3)
$$

provided that the integrals are convergent.

**Acknowledgement**

I am highly thankful to Dr. K. C. Sharma, Reader, Department of Mathematics, University of Rajasthan, Jaipur, for his constant guidance and keen interest during the prepration of this paper.

**References**

1. Bose, S. K. A note on Whittaker transform. *Ganita*, Vol. I (2) : 16–22, (1950).
2. Bose, S. K. On Lapiace transform of two variables, *Bull. Cal. Math. Soc.*, Vol. 41 (4) : (1949).
3. Erdelyi, A. Table of Integral transform, Vol. I : *Mc Graw Hill, New York*, (1954).
4. Kulshreshtha, S. K. On an Integral transform. *Proc. Na'. Acad. Sc., India.* Sec. A 36 (1) : 203–2(8, (1966)
5. Mc Lachlan, N. W. and Humbert, P. Formuiaire pour de calcul symbolique, (1941).
6. Rathie, C. B. Laplace integral and its generalisations. *Thesis approved for Ph.D. decree of Rajasthan University*, (1954).

# Pseudogroup

*By*

R. N. LAL

*Department of Mathematics, Bhagalpur University*

[Received on 20th April, 1966]

**Introduction**

The usual extension of a semi group to a group and our order complete extension of an ordinal semi group[1] are respectively best on both sided cancellation and left cancellation of the underlying operation but John Olmsted[2] has extended a cardinal semi group without cancellation into transfinite rationals, in which division is not the inverse operation of multiplication. Such an extension of Olmsted leads us to a new algebraic structure which may here be called a pseudogroup.

The paper then proposes to develop a theory of a pseudogroup.

**1.** A semi group of idempotents. Let $(S, +)$ be a commutative semi group in which a linear ordering $\leqslant$' is introduced by defining: $a \leqslant b$ provided that there exists an elements $x \leqslant b$ in $S$ such that $a + x = b$. If $x$ is unique, then $x$ is designated as $-a + b$ ; $x$ is called difference $\delta(a,b) = -a + b$ or $-b + a$ according as $a < b$ or $b < a$.

**Theorem 1·1.**

(*i*) A semi group $(S,+)$ consists of idempotents provided that :

$$a \leqslant b \rightleftarrows a + b = b ;$$

(*ii*) in which $a \leqslant b \to a + x = b + x = x$ for $= x \geqslant b$ ;

(*iii*) $\delta(a,b)$ exists if $a \neq b$ ;

(*iv*) the least idempotent $e$ if it exists is a neutral element.

**Proof**

(*i*) $a = a \rightleftarrows a + a = a$

(*ii*) $b \leqslant x. \to . b + x = x$

Also $a \leqslant b, b \leqslant x \to a \leqslant x \to a + x = x .=. b + x.$

(*iii*) Existence of $\delta(b,b) \cdot \to . b + x = b \to x \leqslant b \to$ non uniqueness of $x$.

(*iv*) $e < x \to e + x = . x$ for each $x$.

A semi group with the least idemponent $e$ is a monoid $(S, +, e)$. A bisemi group $(S, +, \cdot)$ consists of common idempotents provided that $a \leqslant b$ iff $a + b = a.b = b$ ; which is called binoid $(S, +, . , e)$ if the least common idempotent $e$ exists.

## 2. Definition of a pseudogroup

*Definition* 2·1 A pseudogroup (shortly p-group) $(G, (-), +)$ is a set $G$ together with a unary $(-)$ and a binary operation $+$ on $G$ satisfying the properties :

$G1$ : $(G, +)$ is a semigroup, *i.e.*, a set $G$ with an associative operation; $(G2)$ for each $a \varepsilon G$, there exists a pseudo inverted element $\bar{a}$ such that $\bar{a} + (a + b) = (a + \bar{a}) + b$ for each $b$; which is a group provided that:

$(G2)^*$ $a + (a + b) = (a + a) + b = b$

Thus a semigroup whose elements are $p$ invertible is called a p-group. Throughout it will be assumed that a p-group is commutative unless otherwise, stated.

*Definition* 2·2. Let $(G, +, (\bar{\ }))$ be a p-group which includes subsets $G^+, G^u$ with the properties :

(1) for each $a \varepsilon G$, either $a \varepsilon G^+$, or $\bar{a} \varepsilon G$, or $a = \bar{a} = a + \bar{a} \varepsilon G^u$

(2) $a, b \varepsilon G^+ \cup G^u .\rightarrow. a + b \varepsilon G^+ \cup G^u$.

Then an element $a$ of $G$ is called positive or negative or an ultra element according as $\bar{a} \varepsilon G^+$ or $\bar{a} \varepsilon G^+$ or $a = \bar{a} \varepsilon G^u$.

In terms of the subsets $G^+$ and $G^u$, it is possible to introduce a partial ordering $\leqslant$ in $G$.

*Definition* 2·3. $a < b$ iff $b + \bar{a} \varepsilon G^+$

$a = b$ iff $\bar{a} + b = a + \bar{b} \varepsilon G^u$

*i.e.*, $a \leqslant b$ iff $\bar{a} + b \varepsilon G^+ \cup G^u$

It can easily be verified that :

$P_1$ : $\leqslant$ partially orders $G$;

$P_2$ : $a, b \varepsilon G^+$ and $a \leqslant b .\rightarrow. a + x = b + x$ for all $x > b$

$P_3$ : $a \leqslant b .\rightleftarrows. \bar{b} \leqslant \bar{a}$.

$P_4$ : If $a \cup b$ exists, then $a \wedge b .=. \overline{(\bar{a} \cup \bar{b})}$ also exists for each $a$ and $b$; ordering '$\leqslant$' is, therefore, a lattice ordering on $G$.

## 3. Embedding

A monoid of idempotents can be extended to a p-group.

**Theorem 3·1.**

A monoid $(S,+,e)$ of idempotents can be embedded into a p-group $(S,+,(\bar{\ }))$ with $e + \bar{e}$ as a neutral element.

**Proof**

Let $E_1$ be an equivalence relation in the cartesian product $S \times S$ defined by the statement :

$$(a_1, b_1)\, E_1 (a_2, b_2) \text{ provided that :}$$

$$m + a_1 \gtreqless n + b_1 \rightleftarrows m + a_2 \gtreqless n + b_2 \text{ for all } m, n \varepsilon s$$

Then there exists a canonical map

$\phi_1$: $S \times S \rightarrow S/E_1$ (quotient set module $E_1$) such that :

$\phi_1(a,b) + \phi_1(c,d) = \phi_1(a+c, b+d)$,

$\phi_1\overline{(a,b)} = \phi_1(b, a)$ and $\phi_1(a,b) - \phi_1(c, d) = \phi_1(a,b) + \phi_1(d,c)$.

Clearly $\phi_1(a,b) + \phi_1(e,e) = \phi_1(a+e, b+e) = \phi_1(a,b)$.

$\phi_1(a,b) \leqslant \phi_1(c,d)$ provided that

$m + a > n + b \rightarrow m + c > m + d$ and $m + a = n + b \rightarrow m + c \geqslant n + d$

$\phi_1(a,b)$ is denoted by $a$-$b$ which is positive or negative or an ultra element and is represented respectively in the canonical form as $a$-$e$ $e$-$b$ or $a$–$a$ according as $a > b$ or $a < b$ or $a = b$.

Clearly $(S/E, +, ^{(-)})$ is a p-group with $e$–$e$ as the neutral element, on which the partial ordering $\leqslant$ is a lattice such that $| a | = a \cup \bar{a}$.

A bisemigroup of common idempotents can be embedded into two p-groups.

**Theorem 3·2.**

A binoid $(S, +, \cdot, e)$ of idempotents can be embedded into a p-group $(S/E, +, ^{(-)})$ with $e$-$e$ as a neutral element on which a lattice ordering can be defined as well as to a multiplicative monoid, positive elements of which are idempotents.

**Proof**

The underlying monoid $(S, +, e)$ of the binoid can be embedded into a p-group $(S/E, +, ^{(-)})$ with a neutral element $e$-$e$. Let us introduce multiplication in the set $(S/E)^{nu} = S/E_1)^+ \cup (\overline{S/E_1})$ of non ultra elements by defining $(a-b)\ (c \cdot d) = \delta(a,b)\ \delta(c,d) - e$ or $e - \delta(a,b)\ \delta(c,d)$, according as the factors are of the same sign or of opposite sign.

Then $(S/E_1)^{nu}, \cdot, e_1 - e)$ is a multiplicative monoid whose positive elements are idempotents with $e_1 - e$ as the least idempotent and which is an extension of the underlying monoid $(S \cdot e, \cdot, e_1)$ of the binoid where $e_1 > e$

**Theorem 3·3.**

The monoid $((S/E_1)^{nu} \cdot, e_1 - e)$ can be embedded into a multiplicative p-group on which a lattice ordering can be defined.

**Proof**

For the sake of brevity the elements of $(S/E_1)^{nu}$ will be denoted by $\alpha, \beta, \gamma \ldots.$ Introduce an equivalence relation $E_2$ in $((S/E_1)^{nu})^2$ by defining : $(\alpha_1, \beta_1)\, E_2\, (\alpha_2, \beta_2)$ iff the couples, have the same sign, or have the opposite sign and further for every $\xi, \eta \varepsilon (S/E_1)^{nu}$, $| \alpha_1 \xi | \gtreqless | \beta_1 \eta | \rightleftarrows | \alpha_2 \xi | \gtreqless | \beta_2 \eta |$

Then there exists a canonical map $\phi_2$ : $( (S/E_1)^{nu} )^2 \rightarrow (S/E_1)/E_2$ such that $\phi_2(\alpha, \beta)\ \phi_2(\gamma, \delta) = \phi_2(\alpha\gamma, \beta\delta)$, $( \phi_2(\alpha, \beta) )^{\dot{-}} = \phi_2(\beta, \alpha)$

Then $(S/E_1)^{nu}/E_2$ is a p-group with $\phi_2(e_1 - e, e_1 - e)$ as a neutral element.

It is possible to introduce a partial ordering in $(S/E_1)/E_2$ which will be a lattice.

**Remark 3·1.**

The above extension of a binoid $(S, +, \cdot, e)$ into a multiplicative group $(S/E_1)^{nu}/E_2$ is obtained *via* the monoid $(S/E_1)^{nu}$ whose positive elements are idempotents.

However, it is possible to extend a binoid into a multiplicative p-group isomorphic to $(S/E)^{nu}/E_2$ directly.

**Theorem 3·4.**

A binoid $S$ can be extended directly into a multiplicative p-group $(S/R)$ isomorphic to the p-group $( (S/E_1)^{nu}/E_2)$.

**Proof**

Let $(S \times S)^* = \{ (b, c) \mid b, c \in s, b \neq c \}$. Introduce an equivalence relation $R$ in the cartesian product $S \times (S \times S)^*$ defined by : $(a ; b, c) \; R \; (d ; e, f)$ iff $b \gtrless c \rightleftarrows e \gtrless . f$ and further $m\, a \gtrless n\, \delta(b,c) \rightleftarrows m\, d \gtrless n\, \delta(e, f)$

Then there exists a canonical map

$\phi_3 : S \times (S \times S)^* \to S/R$ such that :

$\phi_3(a ; b, c)\, \phi_3(d ; e, f) = \phi_3(ad ; \delta(b, c)\, \delta(e, f), 0)$, or

$\phi_3(ad ; 0, \delta(b, c)\, \delta(e, f))$ according as $b \gtrless c \rightleftarrows e \gtrless f$, or $b \gtrless c \rightleftarrows e \lessgtr f$;

$(\phi_3(a ; b, c)^{\dot{-}}$

$= \phi_3(\delta(b, c) ; a, e)$ or $\phi_3(\delta(b, c) ; e, a)$ according as $b > c$ or $b < c$ ; or

$\mid \phi_3 (a ; b, c) \mid \leqslant \mid \phi_3 (d ; e, f) \mid$ iff $m\, a > n\, \delta(b, c) \rightarrow m\, d > n\, \delta(e, f)$

and $ma = n\, \delta(b, c) \rightarrow md \geqslant n\, \delta(e, f)$

$\phi_3(a, b, c) < \phi_3 (d, e, f)$ iff the first is negative and the second is non negative or both are positive and then $\phi_3(a, b, c) \neq \phi_3(d, c, f)$ or both are negative and then $\phi_3(d ; f, e) \leq \phi_3(a ; c, b)$ and $\phi_3(d ; f, e) \neq \phi_3(a ; c\, b$. $\phi_3(a, b, c)$ can be denoted by $a/b - c$ which can be represented in the canonical form as $a/e$ if $b > c$ or $a/a$ if $b > c$ and $a = \delta(b, c)$, or $- a/e$ if $b < c$, or $- a/a$ if $b < c$, and $a = \delta(b, c)$.

It can easily be verified that $(S/R, \cdot, \dot{-})$ is a p-group with $\frac{e_1}{e_1 - e} = \frac{e_1}{e_1}$ as a neutral element on which a lattice ordering can be defined.

The map $\sigma : S/R \to (S/E_1)^{nu}/E_2$ defined by $\sigma(a/b-c) = a/\delta(b,c)$ or $- a/\delta(b, c)$ according as $b > c$ or $b < c$ is an isomorphism of the two p-groups.

## 4. Example

Let $C_\lambda$ be a set of all transfinite cardinals less than the cardinal $w_\lambda$, $+$, $\cdot$, . are usual cardinal operations and $\leqslant$ the usual ordering relation in $C_\lambda$. Then $(C_\lambda, +, \cdot, w)$ is a binoid of idempotents, with $w$ as a neutral element. Then $(C_\lambda/E_1 +, \dot{-})$ is an additive p-group whose elements are called integroids or integroidal numbers of the form $w_\alpha - w_\beta$ which is positive or negative or an ultra element respectively denoted by $w_\alpha - w$, or $w - w_\alpha$ or $w_\alpha - w_\alpha$ according as $w_\alpha > w_\beta$ or $w_\alpha < w_\beta$ or $w_\alpha = w_\beta$

$(C_\lambda/R, \cdot, \dot{-})$ is a p-group whose elements are called rationoidal numbers of the form $w_\alpha/w_\beta - w_\gamma$. A rationoidal number $w_\alpha/w_\beta$ - $w_\gamma$ is positive if $w_\beta > w_\gamma$ and is denoted by $w_\alpha/w$ or $w/w_\alpha$ or $w_\alpha/w_\alpha$ according as $w_\alpha > w_\beta$ or $w_\alpha < w_\beta$ or $w_\alpha = w_\beta$; negative if $w_\beta < w_\gamma$ and is denoted by $-w_\alpha/w$ or $-w/w_\alpha$ or $-w_\alpha/w_\alpha$ according as $w_\alpha > w_\beta$ or $w_\alpha < w_\beta$ or $w_\alpha = w_\beta$.

### References


1. Lal, R. N. A complete extension of ordinal numbers. *Am. Math. Monthly*, **70** : 501-503, (1963).
2. Olmsted, J. Transfinited rationals. *Bull. Am. Math. Soc.* **51** : 776-80, (1945).

# Hyper Real Fields

*By*

R. N. LAL

*Department of Mathematics, University of Bhagalpur*

[Received on 20th April, 1956]

**Introduction**

It is well known that there exists a unique sequentially complete Archimedean ordered field, isomorphic to the ordered field of real numbers. Our extension of ordinal numbers[1] is order complete and contains a complete ordered set of real numbers[2]. It is possible to generalise sequentially completeness and Archimedean ordering in an ordered field and then to break down the uniqueness of a complete ordered field in this generalised sense.

The paper proposes to develop the theory of a $w_\mu$ sequentially complete and $w_\mu$ Archimedean ordered field containing a $w$ sequentially complete Archimedean ordered field. Such a complete ordered field may here be called a hyper real field.

**1. Embedding of an ordered semi-integral domain in an ordered field.**

*Definition* 1·1. An ordered quintiple $< D, +, \cdot, 0, 1 >$ is called a semi-integral domain (SID) provided that :

(*i*) each of the systems $< D, +, 0 >$, $< D, \cdot, 1 >$ is a commutative monoid ;

(*ii*) . is distributed over $+$ ;

(*iii*) $a + b = a + c \rightarrow b = c$ ; which is $w_\mu$ semi integral domain provided that $\text{card}(D) = w_\mu$

*Definition* 1·2. $a < b$ iff $a \varepsilon b$

1·3. A SID is ordered provided that :

$$a < b \rightrightarrows c + a < c + b \text{ for all } c ;$$
$$ca < cb \text{ for all non zero } c.$$

**Theorem 1·1**

$< w_\mu, +_n, \cdot_n, 0, 1 >$ is an ordered SID, where $+_n$ and $\cdot_n$ are natural operations in $w_\mu$.

**Proof**

Let $\alpha \varepsilon w_\mu$ be represented in the summation form $\Sigma w^\lambda m_\lambda$, according to decreasing powers of the base $w$, where $m_\lambda$s are natural numbers.

Then $\Sigma w^\lambda m_\lambda +_n \Sigma w^\lambda p_\lambda = \Sigma w^\lambda (m_\lambda + p_\lambda)$, $\Sigma w^\lambda m_\lambda \cdot_n \Sigma w^\mu p_\mu = \Sigma w^{\lambda +_n \mu} (m_\lambda p_\mu)$

It can easily be verified that $w_\mu$ with natural operations is SID which is ordered with respect to the lexiographic ordering or usual ordering relation.

**Theorem 1·2.**

An ordered $w_{\mu}$ - SID includes :

($i$) a minimal sub domain order isomorphic to $w$ - SID ;

($ii$) subdomains order isomorphic to $w_1, w_2 \ldots\ldots w_{\lambda}$ - SID, where $w_{\lambda} < w_{\mu}$.

The proof is trivial.

**Theorem 1·3.**

$w^{a}$ with natural operations is a SID iff $\alpha$ is an additive indecomposable ordinal $i.e.$ $\alpha$ is not the sum of any two ordinals less than $\alpha$. For $+_n$ is $w^{\alpha} \times w^{\alpha}$ to $w^{a}$ and $\underset{n}{\cdot}$ is $w^{a} \times w^{a}$ to $w^{\alpha}$ iff $\alpha$ is an indecomposable ordinal.

**Theorem 1·4.**

$< w^{u^{\alpha}}, \underset{n}{+}, \underset{n}{\cdot}, 0, 1 >$ is a SID for each ordinal number $\alpha$. For $w^{\alpha}$ is additively indecomposable for each $\alpha$.

**Theorem 1·5.**

An ordered SID can be embedded in an ordered field.

**Proof.**

In a ordered $w$ - SID, $\alpha < \beta \rightarrow \alpha + x = \beta$ for a unique $x \leftarrow D$, where $x = \beta - \alpha$, $D$ then can be embedded in a field by means of triplets. But in an ordered domain such an $x$ does not exist. It is, therefore, natural to extend $D$ in a field by means of quadruplets (qts) as follows :

Let $(D \times D)^* = D^{2*} = \{< \alpha, \beta > \mid \alpha, \beta \leftarrow D, \alpha \neq \beta\} \subset D \times D$. In the cartesian product $D^2 \times D^{2*}$, let $E$ be the equivalence relation defined by the seatement ; $< \alpha, \beta ; \gamma, \delta > E < E\, \xi, \eta ; \zeta, \lambda >$ iff $\alpha\zeta + \beta\sigma + \eta\gamma + \delta\xi = \alpha\sigma + B\zeta + \gamma\xi + \delta\eta$

and let the quotient set $D^2 \times D^{2*}/E$ be denoted by $F$. Then there exists a canonical map $\phi : D^2 \times D^{2*} \rightarrow F_D$ defined by $\phi < \alpha, B ; \gamma, \delta > = \alpha - \beta/\lambda\text{-}\delta$ such that

($i$) $\phi\{ <\alpha, \beta_1;\gamma,\delta_1> + <\alpha_2, \beta_2 ; \gamma_2\delta_2>\} = (\alpha_1\gamma_2 + \beta_1 \delta_2 + \gamma_1\alpha_2 + \delta_1 \beta_2)$ - $(\alpha_1\delta_2 + \beta_1\gamma_2 + \gamma_1\beta_2 + \delta_1\alpha_2) \,/\, (\gamma_1\gamma_2 + \delta_1\delta_2)$ - $(\gamma_1\delta_2 + \delta_1\gamma_2)$.

($ii$) $\phi\{ <\alpha_1,\beta_1 ; \gamma_1,\delta_1> . <\alpha_2, \beta_2 ; \gamma_2\delta_2>\} = (\alpha_1\alpha_2 + \beta_1\beta_2) - (\alpha_1\beta_2 + \alpha_2\beta_1)/ (\gamma_1\gamma_2 + \delta_1\delta_2) - (\gamma_1\delta_2 + \delta_1\gamma_2)$

($iii$) $\phi\, (\alpha,\beta.; \gamma,\delta) < \phi(\xi, \eta ; \zeta, \sigma)$ iff

$\alpha\,\zeta + \beta\sigma + \gamma\eta + \delta\xi < \alpha\sigma + \beta\zeta + \gamma\xi + \delta\eta$

It can easily be verified that : $< F_D. +, \cdot, < >$

is an ordered field containing an ordered :

($i$) SID $\{(D\times 0) \times (1\times 0)\}$ ;

($ii$) integral domain $\phi\, \{(D\times 0) \times (1\times 0)\} \cup \phi\, \{(0\times D) \times (1\times 0)\}$

**Remarks**

1·1. $F_D$ is called the difference quotient field of $D$.

1·2. $F_{w_\mu}$ is a field containing a minimal field $F_w$ order isomorphic to the field of rational numbers $F_{w_\mu}$ includes a subfield $F_{w^{w^\alpha}}$, where $w^{w^\alpha} \leftarrow w_\mu$ for each ordinal $\alpha < w_\mu$ and is, therefore, called a hyper rational field.

1·3. $F_{w_\mu}$ also contains an ordered integral domain whose elements are of the form $\Sigma\, w^\lambda\, (m_\lambda - n_\lambda)$ where $m_\lambda - n_\lambda$ is an integer. Such an integral domain is called the domain of hyper integers.

*Definition* 1·3. An ordering $<$ of an ordered field $F$ is called :

dense in $F$, iff, for any $a,b$ with $a < b$, there is some $c$, *s.t* $a < c < b$ ; $w_\mu$ – Archimedean provided that for each $a \varepsilon F$, there exite an ordinal $\xi \varepsilon w_\mu$ *s.t* $\xi > a$.

Each of the following statements characterizing an $w_\mu$ – Archimedean ordered field is equivalent.

**Theorem 1·6.**

An ordering of an ordered field $F$ is $w_\mu$ – Archimedean iff :

1·6 for each $a \varepsilon F$, there exists an ordinal $\xi \varepsilon w_\mu$ *s.t.* $-\xi < a$ ;

1·7 for each positive $a \varepsilon F$, there exists an ordinal $\xi \varepsilon w_\mu$ *s.t* $1/\xi < a$ ;

1·8 for each pair $a$, $b$ with $a > 0$ acd $a < b$ there exists $a\,\lambda < w_\mu$ *s.t* $\lambda a > b$.

*Definition* 1·4. A subset $X$ is called an open interval iff for some $a$, $b \varepsilon F$, $X = \{ x \mid a < x < b \}$.

## 2. Transfinite sequences in an ordered field.

*Definition* 2·1. A $w_\mu$ sequence of an ordered field $F$ is a map $x : w_\mu \to F$ which is also denoted by $< x_\alpha >$, where $\alpha \varepsilon w_\mu$.

Clearly the ordinary sequence is simply a $w$ – sequence.

*Definition* 2·2 $< x_\alpha >$ is a fundamental sequence in $F$ iff to each $e > 0$, there exists a $\xi_e$ such that $< x_\alpha >$ is in each open interval $< x_\xi - e, x_\xi + c >$ for $\xi > \xi_c$ :

2·3. $< x_\alpha >$ converges to $x$ iff it is eventually in each open interval containing $x$.

In an ordered field $F$ each convergent $w_\mu$ sequence is also a fundamental sequence. But the converse may not hold in $F$, which induces an incompleteness of $F$.

It is to be noted that a $w_\mu$ Archimedean ordered field of hyper rationals is incomplete in any one of the following equivalent senses :

(*i*) There are non convergent $w_\mu$ fundamental sequences in $F_{w_\mu}$ ;

(*ii*) There exists a $w_\mu$ sequence of nested closed intervals not containing a common point.

## 3. Completeness of an ordered field.

Let $S$ be a set of $w_\mu$ fundamental sequences in $F$ and let $E$ be a relation on $S$ defined by : $< x_\alpha > E < y_\alpha >$ iff for each $e > 0$, there corresponds an ordinal number $\xi_e$ s.t. $< y_\alpha >$ is in each open interval $< x_\xi - e, x_\xi + e >$ for $\xi > \xi_e$. The quotient set $S/E$ can easily be verified to be a field with the usual definitions of + and . of sequences. If $S$ be the set of fundamental sequences of hyper rationals, then an element of the quotient set is called a hyper real. The set of hyper reals is denoted by $R$.

*Definition* 3·1. An ordered field $R$ is $w_\mu$ sequentially complete iff such $w_\mu$ fundamental sequence in $R$ is a convergent sequence.

**Theorem 3·2.**

$R$ is a $w_\mu$ sequentially complete ordered field.

Proof is analogous as for $\mu = 0$.

The $w_\mu$ Archimedean of property of $R$ is expressed in the following general form.

**Theorem**

An ordered field $F$ is : $w_\mu$ Archimedean iff ;

3·3 the subset $F_{w_\mu}$ of all hyper rational elements of $F$ is dense in $F$ ;

3·4 each element of $F$ is the accumulation point of a $w_\mu$ sequence of hyper rationals of $F_{w_\mu}$.

Proof is analogous as in the case of $\mu = 0$.

Finally we give a characterization of $R$ among all ordered fields.

The following statements referring to a complete ordered field are equivalent, each characterizing the hyper real field.

**Theorem**

$R$ is : 3·5 $w_\mu$ sequentially complete and $w_\mu$ Archimedean ordered field.

3·6 $w_\mu$ Archimedean and each $w_\mu$ sequence of nested closed intervals in $R$ has a common point.

The proof is analogous, as in the case of ordinary reals.

**References**


1. Lal, R. N. A complete extension of ordinals. *American Mathematical Monthly.*
2. Maximoff, I. On a continum of power $2^{n_1}$, *Annals of Math.*, 41 : 321-27, (1910).

# Laguerre and Hermite Polynomials in Two Variables

*By*

B. P. PARASHAR

*College of Science, Raipur, M. P.*

[Received on 4th June, 1966]

**Abstract**

In this paper the author following Ditkin and Prudnikov, has introduced the Laguerre and Hermite polynomials in two variables in the form :

$$L_n^{(\alpha,\beta)}(x,y) = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n\,!} \sum_{r=0}^{n} \frac{L_{n-r}^{(\alpha)}(x)\,(-y)^r}{r\,!\,\Gamma(\alpha+n-r+1)\,\Gamma(\beta+r+1)}$$

$$H_{2n}(x,y) = \frac{(2n)\,!}{n\,!}\,\pi \sum_{r=0}^{n} \binom{n}{r} \frac{H_{2r}(x)\,y^{2n-2r}}{2^{2r}\,\Gamma(n-r+\frac{1}{2})\,\Gamma(r+\frac{1}{2})}$$

$$H_{2n+1}(x,y) = \frac{(2n+1)\,!\,\pi}{n\,!} \sum_{r=0}^{n} \binom{n}{r} \frac{H_{2r+1}(x)\,y^{2n-2r+1}}{2^{2r+1}\,\Gamma(n-r+\frac{3}{2})\,\Gamma(r+\frac{3}{2})}$$

and has studied certain of their properties. Besides, some definite integrals involving these polynomials have been evaluated by making use of the symbolic calculus of two variables. Incidentally some other interesting results have also been obtained.

## 1. Introduction

Laguerre polynomials have been defined as [5, p. 204]

$$L_n^{(\alpha)}(x) = \frac{e^{x}\,x^{-\alpha}}{n\,!}\,\frac{d^n}{dx^n}\,(e^{-x}\,x^{n+\alpha}) \tag{1·1}$$

with generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\,t^n = (1-t)^{-1-\alpha} \exp\left(\frac{-xt}{1-t}\right) \tag{1·2}$$

while the Hermite polynomials are given by [5, p. 189]

$$H_n(x) = (-1)^n\,e^{x^2}\,\frac{d^n}{dx^n}\,(e^{-x^2}) \tag{1·3}$$

with generating function

$$\sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n\,!} = \exp\,(2xt - t^2) \tag{1·4}$$

The object of this paper is to introduce Laguerre and Hermite polynomials in two variables and to study certain of their properties. Besides, some definite

integrals involving these polynomials have been evaluated by making use of the symbolic calculus of two variables. Incidentally some other interesting results have also been obtained.

**2.** Following [1, p. 139] Laguerre polynomials in two variables may be introduced as follows :

$$(2\cdot1) \qquad L_n^{(\alpha,\,\beta)}(x,\,y) = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n\,!}\sum_{r=0}^{n}\frac{L_{n-r}^{(\alpha)}(x).\,(-y)^r}{r\,!\,\Gamma(\alpha+n-r+1)\,\Gamma(\beta+r+1)}$$

$$Re\ \alpha,\ Re\ \beta > -1$$

Clearly $\qquad L_n^{(\alpha,\beta)}(x,y) = L_n^{(\beta,\alpha)}(y,\,x)$

$$L_n^{(\alpha,\beta)}(x,\,0) = \frac{(1+\beta)_n}{n\,!}\,L_n^{(\alpha)}(x)$$

and $\qquad L_n^{(\alpha,\,0)}(x,\,0) = L_n^{(\alpha)}(x).$

We shall write $\qquad L_n^{(0,\,0)}(x,y) = L_n(x,\,y)$

So that $\qquad L_n(x,y) = L_n(y,\,x) = \sum_{r=0}^{n}\binom{n}{r}\,L_{n-r}^{(x)}\,\frac{(-y)^r}{r\,!}$

with $\qquad L_n(x,\,0) = L_n(x).$

It is easy to derive

$$(2\cdot2) \qquad \sum_{n=0}^{\infty}\frac{n\,!\,L_n^{(\alpha,\beta)}(x,y)\,t^n}{(1+\alpha)_n\,(1+\beta)_n}$$

$$= \Gamma(1+\alpha)\,\Gamma(1+\beta)\,(xt)^{-\alpha/2}\,(yt)^{-\beta/2}.\ e^t J_\alpha(2\sqrt{xt})\,J_\beta(2\sqrt{yt})\ ;\ \alpha,\,\beta > -1$$

of which the corresponding result in one variable [4, (18), p. 189] becomes a special case.

Now using [4, (30) p. 191]

$$\int_0^\infty x^\alpha\,(t-x)^{\beta-1}\,L_n^{(\alpha)}(x)\,dx = \Gamma(\beta)\,\frac{\Gamma(1+\alpha+n)}{\Gamma(1+\alpha+\beta+n)}\,t^{\alpha+\beta}\,L_n^{(\alpha+\beta)}(t)\ ;\ \alpha > -1,\,\beta > 0$$

(2·1) gives

$$(2\cdot3) \qquad \int_0^\infty x^\alpha(t-x)^{\beta-1}\,L_n^{(\alpha,\,\beta)}(x,\,y)\,dx = \Gamma(\beta)\,\frac{\Gamma(1+\alpha+n)}{\Gamma(1+\alpha+\beta+n)}\,t^{\alpha+\beta}$$

$$\times\,L_n^{(\alpha,\,\beta)}(t,y)\ ;\ \alpha > -1,\,\beta > 0.$$

Similarly [3, p. 292]

$$\int_0^\infty x^{\gamma-1} e^{-x} L_n^{(\alpha)} \cdot (x)\, dx = \frac{(\alpha-\gamma+1)_n \Gamma(\beta)}{n!}; \; Re\, \gamma > 0$$

leads to

(2·4)
$$\int_0^\infty x^{\gamma-1} e^{-x} L_n^{(\alpha,\beta)} (x, y)\, dx = \frac{(\alpha-\gamma+1)_n (1+\beta)_n \Gamma(\beta)}{n!\, n!}$$
$$\times\, 2F_2 \left[\begin{matrix} -n, -\alpha-n;\, y \\ 1+\beta, \gamma-\alpha \end{matrix}\right]; \; Re\, \gamma > 0.$$

The Hermite polynomials in two variables may be introduced as below :

(2·5)
$$H_{2n}(x, y) = \frac{(2n)!}{n!} \pi \sum_{r=0}^{n} \binom{n}{r} \frac{H_{2r}(x)\, y^{2n-2r}}{2^{2r}\, \Gamma(n-r+\frac{1}{2})\, \Gamma(r+\frac{1}{2})}$$

$$H_{2n+1}(x, y) = \frac{(2n+1)!\, \pi}{n!} \sum_{r=0}^{n} \binom{n}{r} \frac{H_{2r+1}(x).\, y^{2n-2r+1}}{2^{2r+1}\, \Gamma(n-r+\frac{3}{2})\, \Gamma(r+\frac{3}{2})}$$

Clearly
$$H_{2n}(x, y) = H_{2n}(y, x)$$
$$H_{2n}(x, o) = H_{2n}(x)$$
$$H_{2n+1}(x, y) = H_{2n+1}(y, x)$$

and

$$\left[\frac{1}{2} \frac{H_{2n+1}(x, y)}{y}\right]_{y=0} = H_{2n+1}(x).$$

Also

(2·6)
$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n}(x, y).\, t^{2n}}{(2n)!} = e^{t^2} \cos(2xt) \cos(2yt)$$

and

(2·7)
$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}(x, y)\, t^{2n+2}}{(2n+1)!} = e^{t^2} \sin(2xt) \sin(2yt)$$

With (2·5) and the result [3, p. 288]

$$\int_{-\infty}^{\infty} e^{-x^2} He_{2n}(x)\, dx = (-1)^n \Gamma(n+\tfrac{1}{2})$$

where

$$He_{2n}(x) = \frac{1}{2^n} H_{2n}\left(\frac{x}{\sqrt{2}}\right)$$

we may immediately derive

(2·8)
$$\int_{-\infty}^{\infty} e^{-2x^2} H_{2n}(x, y)\, dx = \sqrt{\frac{\pi}{2}} \frac{1}{2^n} H_{2n}(\sqrt{2}\, y)$$

and

$$(2\cdot 9)\qquad \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} e^{-2(x^2+2y^2)}\, H_{2n}(x, y)\, dx = (-1)^n \tfrac{1}{2} \sqrt{\frac{\pi}{2}}\, \Gamma(n+\tfrac{1}{2})$$

From [3, (5) p. 288, (6), p. 289] by a little change of variable we shall have

$$\int_0^{\infty} e^{-x^2} \sin(\sqrt{2}\,\beta x)\, H_{2n+1}(x)\, dx = (-2)^n \sqrt{\frac{\pi}{2}}\, \beta^{2n+1}.\, e^{-\beta^2/2}$$

and $$\int_0^{\infty} e^{-x^2} \cos(\sqrt{2}\,\beta x)\, H_{2n}(x)\, dx = (-1)^n\, 2^{n-1} \sqrt{\pi}\, \beta^{2n}.\, e^{-\beta^2/2}$$

which yield

$$(2\cdot 10)\qquad \int_0^{\infty} e^{-x^2} \begin{matrix}\sin\\ \sinh\end{matrix} (\sqrt{2}\,\beta x)\, H_{2n+1}(x, y)\, dx = \sqrt{\pi}\, 2^{2n+\frac{1}{2}}.\, \beta.\, e^{-\beta^2/2}.\, y^{2n+1}$$
$$\times\ {}_2F_1\left[\begin{matrix}-n, -n-\tfrac{1}{2};\\ \tfrac{3}{2}\end{matrix}\ \mp \frac{\beta^2}{2y^2}\right]$$

$$(2\cdot 11)\qquad \int_0^{\infty} e^{-x^2} \begin{matrix}\cos\\ \cosh\end{matrix} (\sqrt{2}\,\beta x)\, H_{2n}(x, y)\, dx$$
$$= \frac{\sqrt{\pi}}{2}\, (2y)^{2n}.\, e^{\mp\beta^2/2}\, {}_2F_1\left[\begin{matrix}-n, -n+\tfrac{1}{2};\\ \tfrac{1}{2}\end{matrix}\ \mp \frac{\beta^2}{2y^2}\right] \text{ respectively.}$$

Starting from [4, p. 195]

$$\Gamma(n+\alpha+1) \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}}\, H_{2n}(\sqrt{x}\, t)\, dt$$
$$= (-1)^n \sqrt{\pi}\, (2n)!\, \Gamma(\alpha+\tfrac{1}{2})\, L_n^{(\alpha)}(x)$$

we have

$$(2\cdot 12)\qquad \Gamma(n+\alpha+1) \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}}\, H_{2n}(\sqrt{x}\, t, \sqrt{y})\, dt$$
$$= \frac{(-1)^n\, \pi\, (2n)!\, n!\, \Gamma(\alpha+\frac{1}{2})}{\Gamma(n+\frac{1}{2})}\, L_n^{(\alpha, -\frac{1}{2})}(x, y)$$

Similarly [2, p. 39]

$$\int_0^{\infty} e^{-x^2}\, He_{2n}(2x) \cos(xt)\, dx = \tfrac{1}{2} \sqrt{\pi}\, (-1)^n\, e^{-t^2/2} He_{2n}(t)$$

leads to

$$(2\cdot 13)\qquad \int_0^{\infty} e^{-\frac{x^2}{2}} H_{2n}(x, \sqrt{y}) \cos(x\, u)\, dx$$
$$= \frac{(-1)^n\, (2n)!}{n!}\, \pi \sqrt{\frac{\pi}{2}}\, .\, e^{-u^2} H_{2n}(u, i\sqrt{y}).$$

**3.** The classifical Laplace transform

$$(3\cdot 1) \qquad F(p) = p \int_0^\infty e^{-px} f(x)\, dx$$

which we denote by $F(p) \doteqdot f(x)$ has been generalised in two variables by the Laplace—Carson transfotm[1],

$$(3\cdot 2) \qquad F(p, q) = pq \int_0^\infty \int_0^\infty e^{-px-qy} f(x, y)\, dx\, dy$$

which by analogy with one dimensional symbolism we represent as $F(p, q) \doteqdot f(x, y)$.

$F(p, q)$ is called the image of the function $f(x, y)$.

The images of the polynomials introduced in (2·1) and (2·5) are given by

$$(3\cdot 3) \qquad \frac{(n\,!)^2\, x^\alpha\, y^\beta\, L_n^{(\alpha, \beta)}(x, y)}{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)} \doteqdot \frac{1}{p^\alpha q^\beta}\left(1 - \frac{1}{p} - \frac{1}{q}\right)^n$$

$$Re\ \alpha,\ Re\ \beta > -1$$

$$(3\cdot 4) \qquad \frac{n\,!}{\pi(2n)\,!}\,\frac{H_{2n}(\sqrt{x}, \sqrt{y})}{\sqrt{xy}} \doteqdot \sqrt{pq}\left(\frac{1}{p} + \frac{1}{q} - 1\right)^n$$

$$(3\cdot 5) \qquad \frac{n\,!}{\pi\,(2n+1)\,!}\, H_{2n+1}(\sqrt{x}, \sqrt{y}) \doteqdot \frac{1}{\sqrt{pq}}\left(\frac{1}{p} + \frac{1}{q} - 1\right)^n$$

We shall now evaluate some integrals involving these polynomials by making nse of the symbolic calculus of two variables.

(3·6) The following results[1] shall be required in our investigations

$$(a) \qquad \int_0^x f(x, x-\xi)\, d\xi \doteqdot \frac{F(p, p)}{p} \qquad \text{(p. 47)}$$

$$(b) \qquad \int_0^x \int_0^y f_1(\xi, \eta) . f_2(x-\xi, y-\eta)\, d\xi\, d\eta \doteqdot \frac{F_1(p,q) . F_2(p, q)}{pq} \qquad \text{(p. 43)}$$

$$(c) \qquad \int_0^\infty \int_0^\infty J_0(2\sqrt{sx})\, J_0(2\sqrt{ty})\, f(s, t)\, ds\, dt \doteqdot \frac{1}{pq}\, F\left(\frac{1}{p} + \frac{1}{q}\right) \qquad \text{(p. 58)}$$

$$(d) \qquad \int_0^\infty ds \int_0^x \frac{e^{-\frac{s^2}{4(x-t)}}}{\sqrt{\pi(x-t)}}\, f(s,t)\, dt \doteqdot \frac{F\sqrt{p}, p)}{p} \qquad \text{(p. 66)}$$

$$(e) \qquad \int_0^\infty ds \int_0^x \frac{s . e^{-\frac{s^2}{4(x-t)}}}{2\sqrt{\pi(x-t)^3}}\, f(s, t)\, dt \doteqdot \frac{F(\sqrt{p}, \sqrt{p})}{\sqrt{p}} \qquad \text{(p. 66)}$$

$$(f) \qquad \frac{1}{\pi\sqrt{xy}} \int_0^\infty \int_0^\infty e^{-\frac{s^2}{4x} - \frac{t^2}{4y}} . f(s, t)\, ds\, dt \doteqdot F(\sqrt{p}, \sqrt{q}) \qquad \text{(p. 59)}$$

From [2, p. 174] we shall have

$$\frac{1}{p^{\alpha+\beta+1}}\left(1-\frac{2}{p}\right)^{n} \doteqdot \frac{n\,!}{\Gamma(\alpha+\beta+n+2)}\, x^{\alpha+\beta+1}\, L_n^{(\alpha+\beta+1)}\,(2x)$$

$$Re(\alpha+\beta) > -2$$

If we now apply (3·6 $a$) to (3·3) we have after simplification

$$(3\cdot7) \qquad \int_0^x \frac{\xi^{\alpha}\,(x-\xi)^{\beta}\, L_n^{(\alpha,\,\beta)}\,(\xi,\, x-\xi)\, d\xi}{\Gamma(n+\alpha+1)\;\Gamma(n+\beta+1)} = \frac{x^{\alpha+\beta+1}\, L_n^{(\alpha+\beta+1)}\,(2x)}{n\,!\;\Gamma(\alpha+\beta+n+2)}$$

$$Re(\alpha+\beta) > -2$$

Again from [1, p. 120]

$$\left(1-\frac{1}{p}-\frac{1}{q}\right)^{-1} \doteqdot e^{x+y}\, I_0\,(2\sqrt{xy}\,).$$

Writing

$$\frac{1}{p^{\alpha}\, q^{\beta}}\left(1-\frac{1}{p}-\frac{1}{q}\right) = \frac{\frac{1}{p^{\alpha-1}\, q^{\beta-1}}\left(1-\frac{1}{p}-\frac{1}{q}\right)^{n+1}\cdot\left(1-\frac{1}{p}-\frac{1}{q}\right)^{-1}}{pq}$$

and using (3·6 $b$) we get

$$(3\cdot8) \qquad \int_0^x\int_0^y \xi^{\alpha-1}\,\eta^{\beta-1}\, L_{n+1}^{(\alpha-1,\,\beta-1)}\,(\xi,\,\eta).\; e^{x+y-\xi-\eta}\, I_0(2\sqrt{(x-\xi)\;(y-\eta)})\; d\xi\; d\eta$$

$$= \frac{x^{\alpha}\, y^{\beta}\, L_n^{(\alpha,\,\beta)}\,(x,\,y)}{(n+1)^{2}}$$

Similarly writing

$$\frac{1}{\sqrt{pq}}\left(\frac{1}{p}+\frac{1}{q}-1\right)^{n} = -\frac{\sqrt{pq}\,\left(\frac{1}{p}+\frac{1}{q}-1\right)^{n+1}}{pq} \times \left(1-\frac{1}{p}-\frac{1}{q}\right)^{-1}$$

and using (3·6, $b$) we have

$$(3\cdot9) \qquad \int_0^x\int_0^y \frac{H_{2n+2}(\sqrt{\xi},\sqrt{\eta})}{\sqrt{\xi\,\eta}}\, e^{x+y-\xi-\eta}\; I_0\;(2\sqrt{(x-\xi)\;(y-\eta)}\,)\; d\xi\; d\eta$$

$$= -\,2H_{2n+1}(\sqrt{x},\sqrt{y}\,)$$

If we write

$$\frac{1}{\sqrt{pq}}\left(\frac{1}{p}+\frac{1}{q}-1\right)^{n} = \frac{(-1)^{n}\,\sqrt{pq}\;.\;\left(1-\frac{1}{p}-\frac{1}{q}\right)^{n}}{pq}$$

it follows that

$$(3\cdot10) \qquad \int_0^x\int_0^y \frac{1}{\sqrt{\xi\eta}}\, L_n(x-\xi,\, y-\eta)\; d\xi\; d\eta = \frac{(-1)^{n}\, n\,!}{(2n+1)\,!}\, H_{2n+1}\,(\sqrt{x},\sqrt{y}\,)$$

Application of (3·6 $a$) to (3·4) with

$$L_n(2x) \doteqdot \left(1-\frac{2}{p}\right)^n$$

gives

$$(3\cdot 11) \qquad \frac{n!}{\pi(2n+1)!}\int_0^x \frac{H_{2n}(\sqrt{x}, \sqrt{x-\xi})}{\sqrt{x(x-\xi)}}\, d\xi = (-1)^n L_n(2x)$$

Next from (3·6 $a$) and (3·5) and

$$\frac{(n+2)(n+1)}{p^2}\left(1-\frac{2}{p}\right)^n \doteqdot x^2 L_n^{(2)}(2x)$$

we get

$$(3\cdot 12) \qquad \frac{(n+2)!}{\pi(2n+1)!}\int_0^x H_{2n+1}(\sqrt{x}, \sqrt{x-\xi})\, d\xi = (-1)^n x^2 L_n^{(2)}(2x)$$

**4.** It is worthwhile to observe that some interesting results are obtained by the application of the operational calculus of two variables on simple functions.

For example, starting from the relations [1, p. 106, 107]

$$\sinh(x+y) \doteqdot \frac{pq(p+q)}{(p^2-1)(q^2-1)}$$

$$\cosh(x+y) \doteqdot \frac{pq(pq+1)}{(p^2-1)(q^2-1)}$$

$$\cos(x+y) \doteqdot \frac{pq(pq-1)}{(p^2-1)(q^2-1)}$$

and using (3·6 $c$) we get

$$(4\cdot 1) \qquad \int_0^\infty\int_0^\infty J_0(2\sqrt{sx})\,J_0(2\sqrt{ty}) \begin{matrix}\sinh\\ \cosh\\ \cos\end{matrix}(s+t)\,ds\,dt = \begin{matrix}\sinh\\ \cosh\\ -\cos\end{matrix}(x+y)$$

Further from [2, p. 266]

$$erf\left(\frac{x}{2}\right) \doteqdot e^{p^2}\, erfc(p)$$

and from [1, p. 155]

$$\frac{1}{\sqrt{\pi}} J_0(x\sqrt{y}) \doteqdot p\sqrt{q}\, e^{p^2 q}\, erfc(p\sqrt{q})$$

Now applying (3·6 $d$) we get

$$(4\cdot 2) \qquad \int_0^\infty ds \int_0^x \frac{e^{-\frac{s^2}{4(x-t)}}}{\pi\sqrt{(x-t)}} J_0(s\sqrt{t})\, dt = erf\left(\frac{x}{2}\right)$$

From [2, p. 176]

$$-i\, e^{-x}\, erf(i\sqrt{x}) \doteqdot \frac{\sqrt{p}}{p+1}$$

and from [1, p. 137]

$$J_0(2\sqrt{xy}) \doteqdot \frac{pq}{pq+1}$$

Application of (3·6 $e$) leads to

$$(4\cdot3) \qquad \int_0^\infty ds \int_0^x \frac{s \cdot e^{-\frac{s}{4(x-t)}}}{2\sqrt{\pi(x-t)^3}} J_0(2\sqrt{st}\,)\, dt = -\, i\, e^{-x}\, erf(i\sqrt{x}\,)$$

Lastly, making use of the relations [1, p. 134, 135]

$$\frac{1}{a^n}\left(\frac{x}{y}\right)^{n/2} I_n(2a\sqrt{xy}\,) \doteqdot \frac{pq}{p^n\,(pq-a^2)}$$

$$\frac{1}{2\,a^n}\left(\frac{x}{y}\right)^n [J_{2n}(2\sqrt{axy}) + I_{2n}(2\sqrt{axy})] \doteqdot \frac{p^2\, a^2}{p^{2n}\,(p^2\, q^2 - a^2)}\ ; \ n > -\tfrac{1}{2}$$

and applying (3·6 $f$) we have

$$(4\cdot4) \qquad \frac{1}{2\pi\sqrt{xy}} \int_0^\infty\int_0^\infty e^{-\frac{s^2}{4x}-\frac{t^2}{4y}} \left[\left(\frac{s}{t}\right)^n \{J_{2n}(2\sqrt{ast}) + I_{2n}(2\sqrt{ast})\}\right] ds\, dt$$

$$= \left(\frac{x}{y}\right)^{n/2} I_n(2a\sqrt{xy}\,)\ ; \ n > -\tfrac{1}{2}$$

**Acknowledgement**

The author expresses his gratitude to Dr. C. B. L. Verma for his guidance in the preparation of this paper.

**References**

1. Ditkin, V. A. and Prudnikov, A. P. Operational calculus in Two Variables and its Applications. *Pergamon Press*, (1962).
2. Erdelyi, A. *Tables of Integral Transforms*, Vol. 1, (1954).
3. Erdelyi, A. *Tables of Integral Transforms* Vol. 2, (1954).
4. Erdelyi, A. *Higher Transcendental Functions*, Vol. 2, (1954).
5. Rainville, E. D. *Special Functions* (1960).

# On Gauss's Hypergeometric Transform

*By*

K. C. GUPTA

*Department of Mathematics, M. R. Engineering College, Jaipur*

*and*

S. S. MITTAL

*Department of Mathematics, University of Rajasthan, Jaipur*

[Received on 14th May, 1966]

**Abstract**

The aim of this note is to establish a theorem involving Gauss's Hypergeometric function transform and generalized Laplace transform. An integral involving product of generalized Hypergeometric functions given by Maitland Wright has been evaluated as an application of the theorem.

## 1. Introduction

Recently Rajendra Swaroop [5, p. 107] has obtained an inversion formula and uniqueness theorem for Gauss's Hypergeometric function transform defined below :

$$G\{f(x)\,;k,r\,;\eta\,;s\}$$
$$=\frac{\Gamma(k)\,\Gamma(r)}{\Gamma(\eta)}\int_0^\infty F\left(\begin{matrix}k, r\\ \eta\end{matrix}\,;\,-\frac{x}{s}\right)f(x)\,dx \tag{1·1}$$

when $r=\eta$ (1·1) yields

$$S\{f(x)\,;k\,;s\}=\frac{s^{1-k}}{\Gamma(k)}\left[G\{f(x)\,;k,r\,;\eta\,;s\}_{\eta\,=\,r}\right] \tag{1·2}$$

$$=s\int_0^\infty f(x)\,(s+x)^{-k}\,dx \tag{1·3}$$

We shall call $S\{f(x)\,;k\,;s\}$, a generalized Stieltjes transform.

The aim of this paper is to establish an interesting theorem involving the Gauss's Hypergeometric transform defined by (1·1) and the following generalized Laplace transform given by Mainra [4, p. 23].

$$W\{f(x)\,;\eta'+\tfrac{1}{2}\,;k'+\tfrac{1}{2},r'\,;s\}$$
$$=s\int_0^\infty (s\,x)^{-\eta'-\frac{1}{2}}\,e^{-\frac{1}{2}sx}\,W_{k'+\frac{1}{2},r'}(sx)\,f(x)\,dx \tag{1·4}$$

An interesting integral involving product of generalized hypergeometric functions given by Maitland [7, p. 287] has been evaluated as an application to the theorem. Two theorems recently obtained by Gupta [2 ; 6, p. 710] follow as particular cases of our theorem.

## 2. The H-function

The H-function introduced by Fox [1, p. 408], will be represented and defined as follows :

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),(b_2,\beta_2),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$\frac{1}{2\pi i}\int_L \frac{\prod\limits_{j=1}^{m}\Gamma(b_j-\beta_j\,\xi)\prod\limits_{j=1}^{n}\Gamma(1-a_j+\alpha_j\,\xi)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\,\xi)\prod\limits_{j=n+1}^{p}\Gamma(a_j-\alpha_j\,\xi)}\,x^{\xi}\,d\xi, \tag{2·1}$$

where $x$ is not equal to zero and an empty product is interpreted as unity ; $p, q, m, n$ are integers satisfying $1 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$ ; $\alpha_j$ $(j = 1, \ldots, p)$, $\beta_j$ $(j = 1, \ldots, q)$, are positive numbers and $a_j$ $(j = 1, \ldots, p)$, $b_j$ $(j = 1 \ldots, q)$, are complex numbers such that no pole of $\Gamma(b_h - \beta_h\,\xi)$ $(h = 1, \ldots, m)$, coincides with any pole of $\Gamma(1 - a_i + \alpha_i\,\xi)$ $(i = 1, \ldots, n)$, *i.e.*,

$$\alpha_i\,(b_h + \nu) \neq (a_i - \eta - 1)\,\beta_h \tag{2·2}$$

$(\nu, \eta = 0, 1, \ldots ; h = 1, \ldots, m ; i = 1, \ldots, n)$

Further the contour $L$ runs from $\sigma - i\infty$ to $\sigma + i\infty$ such that the points :

$$\xi = (b_h + \nu)/\beta_h\ (h = 1, \ldots, m ; \nu = 0, 1, \ldots ;) \tag{2·3}$$

which are poles of $\Gamma(b_h - \beta_h\,\xi)$, lie to the right and the points :

$$\xi = (a_i - \eta - 1)/\alpha_i\ (i = 1, \ldots, n ; \eta = 0, 1, \ldots ;) \tag{2·4}$$

which are poles of $\Gamma(1 - a_i + \alpha_i\,\xi)$, lie to the left of $L$. Such a contour is possible on account of (2·2). These assumptions for the H-function will be adhered to throughout this paper.

### Properties of the H-function

The H-function is symmetric in the pairs $(a_1, \alpha_1), (a_2, \alpha_2), \ldots, (a_n, \alpha_n)$, likewise in $(a_{n+1}, \alpha_{n+1}), \ldots, (a_p, \alpha_p)$, in $(b_1, \beta_1), \ldots, (b_m, \beta_m)$, and in $(b_{m+1}, \beta_{m+1}), \ldots, (b_q, \beta_q)$.

If one of $(a_i, \alpha_i)$ $(i = 1, \ldots, n)$, is equal to one of $(b_j, \beta_j)$ $(j = m + 1, \ldots q)$ [or one of the $(b_h, \beta_h)$ $(h = 1, \ldots, m)$ is equal to one of the $(a_j, \alpha_j)$ $(j = n + 1, \ldots, p)$] , then the H-function reduces to one of the lower order, that is $p$, $q$, and $n$ (or $m$) decreases by unity ; we give below one of such reduction formulae :

(a)
$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),(b_2,\beta_2),\ldots,(b_{q-1},\beta_{q-1}),(a_1,\alpha_1)\end{matrix}\right.\right]$$

$$= H_{p-1,q-1}^{m,n-1}\left[x\left|\begin{matrix}(a_2,\alpha_2),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q-1,\beta_q-1)\end{matrix}\right.\right], \tag{2·5}$$

other reduction formulae being similar.

The obvious changes of the variable in the integral of right hand side of (2·7) give us the following result :

$$(b) \qquad H_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]$$

$$= c\, H_{p,\,q}^{m,\,n}\left[x^c \,\middle|\, \begin{matrix}(a_1, c\alpha_1), \ldots, (a_p, c\alpha_p)\\ (b_1, c\beta_1), \ldots, (b_q, c\beta_q)\end{matrix}\right] \qquad (2{\cdot}6)$$

where $c > 0$.

The following particular cases of the H-function have been pointed out recently in an earlier paper in these proceedings[3]

$$(1) \qquad H_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix}(a_1, 1), \ldots, (a_p, 1)\\ (b_1, 1), \ldots, (b_q, 1)\end{matrix}\right] = G_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right] \qquad (2{\cdot}7)$$

where the function on the right hand side of (2·7) denotes the well known Meijer's G-function.

$$(2) \qquad H_{p+2,\,q+1}^{m,\,n+2}\left[x \,\middle|\, \begin{matrix}(l_1, S), (l_2, S), (a_1, N), \ldots, (a_p, N)\\ (b_1, N), \ldots, b_q, N), (f, S)\end{matrix}\right]$$

$$= S^{\frac{1}{2}+f-l_1-l_2}\, N^{\sum_{j=1}^{q}(b_j) - \sum_{j=1}^{p}(a_j) + \frac{1}{2}p - \frac{1}{2}q}\, (2\pi)^{\frac{1}{2}(1-s)+(1-N)(m+n-\frac{1}{2}p-\frac{1}{2}q)}\,.$$

$$\times\, G_{2S+Np,\,Nq+S}^{Nm,\,2S+Nn}\left[x\, S^s\, N^{N(p-q)} \,\middle|\, \begin{matrix}\Delta(S,l_1), \Delta(S,l_2), \Delta(N,a_1), \ldots, \Delta(N,a_p)\\ \Delta(N, b_1), \ldots, \Delta(N, b_q), \Delta(S,f)\end{matrix}\right] \qquad (2{\cdot}8)$$

Where throughout this paper $\Delta\,(S, a)$ stands for the parameters

$$\frac{a}{S}, \frac{a+1}{S}, \frac{a+2}{S}, \ldots, \frac{a+S-1}{S}$$ and $N$, $S$ for positive integers.

(3) The function defined by the series :

$$\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + \alpha_j r)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j r)} \frac{(-x)^r}{\lfloor r}$$

was considered in detail by Maitland [7, p. 287]. We shall call this function as Maitland's generalized hypergeometric function and denote it symbolically as :

$${}_p\psi_q \left[\begin{matrix}(a_1; \alpha_1), \ldots, (a_p, \alpha_p)\\ (b_1\, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\; ; -x\right]$$

The following formula gives us the relationship between this function and the H-function :

$$ {}_p\psi_q \left[ \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} ; -x \right] $$

$$ = H^{1, p}_{p, q+1} \left[ x \middle| \begin{matrix} (1-a_1, \alpha_1), (1-a_2, \alpha_2), \ldots, (1-a_p, \alpha_p) \\ (0, 1), (1-b_1, \beta_1), \ldots, (1-b_q, \beta_q) \end{matrix} \right] \tag{2·9} $$

**3.** The following results[3] will be used in our subsequent discussions later on :

$$ (a) \quad \frac{\Gamma(k)\,\Gamma(r)}{\Gamma_\eta} \; W \left\{ x^l \; F(k, r ; \eta ; -\frac{x^\sigma}{a}) ; \eta' + \tfrac{1}{2} ; k' + \tfrac{1}{2}, r' ; s \right\} $$

$$ = s^{-l} \; {}_4\psi_2 \left[ \begin{matrix} (1 + l - \eta' \pm r', \sigma), (k, 1), (r, 1) \\ (\eta, 1), (1 + l - \eta' - k', \sigma) \end{matrix} ; -\frac{s^{-\sigma}}{a} \right] \tag{3·1} $$

where $\sigma > 0$, $R(s) > 0$, $R(l + 1 - \eta' \pm r') > 0$ and $| \arg a | < \pi$.

$$ (b) \quad W \left\{ {}_p\psi_q \left[ \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} ; -zx \right] ; \eta' + \tfrac{1}{2} ; k' + \tfrac{1}{2}, r' ; s \right\} $$

$$ = H^{1, p+2}_{p+2, q+2} \left[ \frac{z}{s} \middle| \begin{matrix} (\eta' \pm r', 1), (1-a_1, \alpha_1), \ldots, (1-a_p, \alpha_p) \\ (0, 1), (1-b_1, \beta_1), \ldots, (1-b_q, \beta_q), (\eta' + k', 1) \end{matrix} \right] \tag{3·2} $$

where $\sigma > 0$, $R(s) > 0$, $R(l + 1 - \eta' \pm r') > 0$, $1 + \sum_{j=1}^{p} (\alpha_j) - \sum_{j=1}^{q} (\beta_j) > 0$,

and $$ | \arg z | < \left( 1 + \sum_{j=1}^{p} (\alpha_j) - \sum_{j=1}^{q} (\beta_j) \right) \frac{\pi}{2} $$

$$ (c) \quad \int_0^\infty x^{l-1} \; H^{m, n}_{p, q} \left[ z x^\sigma \middle| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right] {}_2F_1 (c_1, c_2 ; d ; -s x) \, dx $$

$$ = \frac{s^{-l}\,\Gamma(d)}{\Gamma(c_1)\,\Gamma(c_2)} H^{m+2, n+1}_{p+2, q+2} \left[ \frac{z}{s^\sigma} \middle| \begin{matrix} (a_1, \alpha_1), \ldots, (a_n, \alpha_n), (1-l, \sigma), (d-l, \sigma), (a_{n+1}, \alpha_{n+1}), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_m, \beta_m), (c_1 - l, \sigma), (c_2 - l, \sigma), (b_{m+1}, \beta_{m+1}), \ldots, (b_q, \beta_q) \end{matrix} \right] \tag{3·3} $$

where $R(l + \sigma \min \frac{b_h}{\beta_h}) > 0$ $(h = 1, \ldots, m)$ $\sigma > 0$,

$$ R \left\{ l + \sigma \max \left( \frac{a_i - 1}{\alpha_i} \right) - \min c_j \right\} < 0 \; (i = 1, \ldots, n ; j = 1,2) \; | \arg s | < \pi, $$

$$ \lambda = \sum_{1}^{n} (\alpha_j) - \sum_{n+1}^{p} (\alpha_j) + \sum_{j=1}^{m} (\beta_j) - \sum_{m+1}^{q} (\beta_j) > 0 \text{ and } | \arg z | < \tfrac{1}{2} \lambda \pi. $$

## 4. Theorem

If $\quad W\{\phi(x)\,;\,\eta' + \frac{1}{2}\,;\,k' + \frac{1}{2},\,r'\,;\,s\} = s^{\rho} f(s^{\sigma}),$ (4·1)

and
$$G\{x^{\frac{\rho+l}{\sigma}-1} f(x)\,;\,k,\,r\,;\,\eta\,;\,s\} = h(s)$$

then

$$h(s) = \sigma \int_0^{\infty} x^{-l-1}\,{}_4\psi_2 \left[ \begin{matrix} (1-\eta'+l \pm r',\sigma),\,(k,1),\,(r,1) \\ (\eta,1),\,(1+l-k'-\eta',\sigma) \end{matrix}\,;\,-\frac{x^{-\sigma}}{s} \right] \phi(x)\,dx \quad (4·2)$$

provided Whittaker transform of $|\,\phi(x)\,|$ and Gauss's hypergeometric transform of $|\,x^{\frac{\rho+l}{\sigma}-1} f(x)\,|$ exist and the integral given by (4·2) is convergent.

**Proof**

Mainra [4, p. 25] has proved a theorem for the transform given by (1·4) which is analogous to Parseval Goldstein theorem for Laplace transform : Mainra's theorem states that :

If
$$W\{\phi_1(x)\,;\,\eta' + \tfrac{1}{2}\,;\,k' + \tfrac{1}{2},\,r'\,;\,s\} = f_1(s)$$

and
$$W\{\phi_2(x)\,;\,\eta' + \tfrac{1}{2}\,;\,k' + \tfrac{1}{2},\,r'\,;\,s\} = f_2(s)$$

then
$$\int_0^{\infty} \frac{\phi_1(x) f_2(x)}{x}\,dx = \int_0^{\infty} \frac{\phi_2(x) f_1(x)}{x}\,dx \quad (4·3)$$

provided that one of the integrals in (4·3) is absolutely convergent and Mainra transforms of $|\,\phi_1(x)\,|$ and $|\,\phi_2(x)\,|$ exist. Using this theorem in the pairs given by (3·1) and (4·1) we get :

$$\frac{\Gamma(k)\,\Gamma(r)}{\Gamma(\eta)} \int_0^{\infty} x^{l+\rho-1} f(x^{\sigma})\; F\left( k,\,r\,;\,\eta\,;\,-\frac{x^{\sigma}}{a} \right) dx$$
$$= \int_0^{\infty} x^{-l-1}\,\phi(x)\,{}_4\psi_2 \left[ \begin{matrix} (1+l-\eta' \pm r',\sigma),\,k,1),\,(r,1) \\ (\eta,1),\,(1-\eta'+l-k',\sigma) \end{matrix}\,;\,-\frac{x^{-\sigma}}{a} \right] dx \quad (4·4)$$

On replacing $a$ by $s$ in (4·4), we get the required result after a little simplification

**Corollary 1.**

On taking $\eta' = k'$ and $k' = \pm r'$ in theorem 1, we get.

If
$$L\,\{\phi(x)\,;\,s\} = s^{\rho} f(s^{\sigma})$$

and
$$G\{x^{\frac{\rho+l}{\sigma}-1} f(x)\,;\,k,\,r\,;\,\eta\,;\,s\} = h(s)$$

then

$$h(s) = \sigma \int_0^\infty x^{-l-1} \; {}_3\psi_1 \left[ \begin{matrix} (1 + l, \sigma), (k, 1), (r, 1) \\ (\eta, 1) \end{matrix} ; - \frac{x^{-\sigma}}{s} \right] \phi(x) \, dx \tag{4·5}$$

provided that Laplace's transform of $| \phi(x) |$ and Gauss's Hypergeometric function transform of $| x^{\frac{\rho+l}{\sigma} - 1} f(x) |$ exist and the integral given by (4·5) is absolutely convergent.

**Corollary 2.**

If we put $\eta = r$ in the theorem given above, we obtain by virtue of (1·2) and (1·3) the following corollary :

If

$$W\{\phi(x) \, ; \eta' + \tfrac{1}{2} \, ; k' + \tfrac{1}{2}, r' \, ; s\} = s^\rho f(s^\sigma)$$

then

$$S\{x^{\frac{\rho+l}{\sigma} - 1} f(x) \, ; k \, ; s\} = h(s)$$

and

$$h(s) = \frac{s^{1-k}}{\Gamma(k)} \sigma \int_0^\infty x^{-l-1} \phi(x) \, {}_3\psi_1 \left[ \begin{matrix} (1 - \eta' + l \pm r', \sigma), (k, 1) \\ (1 - \eta' - k' + l, \sigma) \end{matrix} \; ; \frac{x^{-\sigma}}{s} \right] dx \tag{4·6}$$

provided that Mainra transform of $| \phi(x) |$ and generalized Stieltjes transform of $| x^{\frac{\rho+l}{\sigma} - 1} f(x) |$ exist and the integral given by (4·6) is absolutely convergent.

(4·6) was given by Gupta[2] in a slightly different form.

**Corollary 3.**

If we take $\sigma = N/S$, $\eta' = -r'$ and replace $k'$ by $k' - \frac{1}{2}$ in corollary 2 we get a known theorem [6, p. 710] by virtue of the relation (2·5), (2·6) and (2·8).

**Example.**

Taking

$$\phi(x) = {}_p\psi_q \left[ \begin{matrix} (a_1, \alpha_1), (a_2, \alpha_2), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \; ; -zx \right]$$

in theorem 1 we get by virtue of (3·2), under the conditions stated there :

$$s^\rho f(s^\sigma) = H^{\,1, p+2}_{\,p+2, q+2} \left[ \frac{z}{s} \middle| \begin{matrix} (\eta' \pm r', 1), (1 - a_1, \alpha_1), \ldots, (1 - a_p, \alpha_p) \\ (0, 1), (1 - b_1, \beta_1), \ldots, (1 - b_q, \beta_q), (\eta' + k', 1) \end{matrix} \right]$$

$$\therefore x^{\frac{l+\rho}{\sigma} - 1} f(x) = x^{l/\sigma - 1} H^{\,1, p+2}_{\,p+2, q+2} \left[ z x^{-1/\sigma} \middle| \begin{matrix} (\eta' \pm r', 1), (1 - a_1, \alpha_1), \ldots, (1 - a_p, \alpha_p) \\ (0, 1), (1 - b_1, \beta_1), \ldots, (1 - b_q, \beta_q), (\eta' + k', 1) \end{matrix} \right]$$

On finding the Gauss's Hypergeometric transform of $x^{\frac{\rho+l}{\sigma}-1} f(x)$ with the help of (3·3), we get under the conditions directly obtainable from there

$$h(s)=s^{l/\sigma} H \begin{matrix} p+4,\, 2 \\ q+4,\, p+4 \end{matrix} \left[ \frac{s^{1/\sigma}}{z} \middle| \begin{matrix} (1,1),\, (1-l/\sigma,\, 1/\sigma),\, (\eta-l/\sigma,\, 1/\sigma), \\ (1-\eta' \pm r',\, 1),\, (a_1,\, \alpha_1),\, \ldots, \end{matrix} \right.$$

$$\left. \begin{matrix} (b_1,\, \beta_1),\, \ldots,\, (b_q,\, \beta_q),\, (1-\eta'-k',\, 1) \\ (a_p,\, \alpha_p),\, (k-l/\sigma,\, 1/\sigma),\, (r-l/\sigma,\, 1/\sigma) \end{matrix} \right]$$

Using these values of $\phi(x)$ and $h(s)$ in (4·2) we get the following integral :

$$\int_0^\infty x^{-l-1}\, {}_4\psi_2 \left[ \begin{matrix} (1+l-\eta' \pm r',\, \sigma),\, (k,\, 1),\, (r,\, 1) \\ (\eta,\, 1),\, (1+l-\eta'-k',\, \sigma) \end{matrix} ; -1/s\, x^{-\sigma} \right]$$

$$\times\, {}_p\psi_q \left[ \begin{matrix} (a_1,\, \alpha_1),\, \ldots,\, (a_p,\, \alpha_p) \\ (b_1,\, \beta_1),\, \ldots,\, (b_q,\, \beta_q) \end{matrix} ; -zx \right] dx$$

$$= 1/\sigma\, s^{l/\sigma} H \begin{matrix} p+4,\, 2 \\ q+4,\, p+4 \end{matrix} \left[ \frac{s^{1/\sigma}}{z} \middle| \begin{matrix} (1,\, 1),\, (1-l/\sigma,\, 1/\sigma),\, (\eta-l/\sigma,\, 1/\sigma), \\ (1-\eta' \pm r',\, 1),\, (a_1,\, \alpha_1),\, \ldots, \end{matrix} \right.$$

$$\left. \begin{matrix} (b_1,\, \beta_1),\, \ldots,\, (b_q,\, \beta_q),\, (1-\eta'-k',\, 1) \\ (a_p,\, \alpha_p),\, (k-l/\sigma,\, 1/\sigma),\, (r-l/\sigma,\, (1/\sigma) \end{matrix} \right] \qquad (4{\cdot}7)$$

where $\sigma > 0,\, |\arg s| < \pi,\, 1 + \sum_{j=1}^{p} (\alpha_j) - \sum_{j=1}^{q} (\beta_j) > 0$

$$|\arg z| < \left(1 + \sum_{j=1}^{p} (\alpha_j) - \sum_{j=1}^{q} (\beta_j)\right)\frac{\pi}{2},\ R\left(l + \frac{a_i}{\alpha_i}\right) > 0\ (i = 1, \ldots, p),$$

$R(1-\eta' \pm r') > 0$, $R(\sigma k - l) > 0$ and $R(\sigma r - l) > 0$.

**References**


1. Fox, C. G and H function as symmetrical Fourier Kernels. *Trans. Amer. Math. Soc.* 98 : 408, (1961).
2. Gupta, K. C. Thesis approved for the award of Ph.D. degree. *Univ. of Rajasthan Jaipur*, (1966).
3. Gupta, K. C. and Jain, U. C. The H-function II. *Proc. Nat. Acad. Sci. India (In press)*.
4. Mainra, V. P. A new generalization of the Laplace transform. *Bull. Calcutta, Math. Soc.* 53 : 23–25, (1961).
5. Rajendra Swaroop. On a generalization of the Laplace and the Stieltjes transformation *Annales de la Societe Scientifique de Bruxelles* 78 : 107, (1964).
6. Saxena, R. K. and Gupta, K. C. Certain properties of generalized Stieltjes transform involving Meijer's G function. *Proc. Nat. Inst Sci. India*, 30 A : 710, (1964).
7. Wright, E. M. The asymptotic expansion of the generalized Hypergeometric function. *Journ. London. Math. Soc.* 10 : 287, (1935).

# On the Derivatives of the H function

By

A. N. GOYAL and G. K. GOYAL

*Department of Mathematics, University of Rajasthan, Jaipur*

[Received on 4th June, 1966]

**Abstract**

In this paper we have given the nth. derivatives of the H-function. Certain recurrence relations are obtained by putting $n=1$. Bhise[2] results follow as particular cases.

**Introduction**

Fox's[1] H-function is defined as

$$H_{p,\,q}^{1,\,u}\left[x\,\middle|\,\begin{matrix}(a_1, e_1)\ \ldots\ldots\ (a_p, e_p)\\(b_1, f_1)\ \ldots\ldots\ (b_q, f_q)\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_L \frac{\prod\limits_{j=1}^{l}\Gamma(b_j-f_j\,s)\prod\limits_{j=1}^{u}\Gamma(1-a_j+e_j\,s)}{\prod\limits_{j=l+1}^{q}\Gamma(1-b_j+f_j\,s)\prod\limits_{j=u+1}^{p}\Gamma(a_j-e_j\,s)}\,x^s\,ds \tag{1·0}$$

where an empty product is interpreted as 1, $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; $e$'s and $f$'s are all positive; $L$ is a suitable contour of Branes type such that the poles of $\Gamma(b_j - f_j s)$, $j = 1, 2, \ldots\ldots 1$ lie on the right hand side of the contour and those of $\Gamma(1 - a_j + e_j\, s)$, $j = 1, 2 \ldots\ldots u$ lie on the left hand side of the contour. Also the parameters are so restricted that the integral on the right hand side of (1·0) is convergent.

Consider.*

$$\frac{d^n}{dx^n}\left[\ x^{a_1-1}\,H_{p\,q}^{l\,u}\left(\frac{\beta}{x^{e_1}}\,\middle|\,\begin{matrix}(a_1, e_1)\ \ldots\ (a_p, e_p)\\(b_1, f_1)\ \ldots\ (b_q, f_q)\end{matrix}\right)\right]$$

$$=(-1)^n\,x^{a_1-n-1}\,H_{pq}^{l\,u}\left[\frac{\beta}{x^{e_1}}\,\middle|\,\begin{matrix}(a_1-n, e_1), (a_2, e_2)\ \ldots\ (a_p, e_p)\\(b_1, f_1)\ \ldots\ (b_q, f_q)\end{matrix}\right] \tag{1·1}$$

To prove (1·1) we have from (1·0)

$$\frac{d^n}{dx^n}\left[x^{a_1-1}\,H_{pq}^{l\,u}\left(\frac{\beta}{x^{e_1}}\,\middle|\,\begin{matrix}(a_1, e_1)\ \ldots\ (a_p, e_p)\\(b_1, f_1)\ \ldots\ (b_q, f_q)\end{matrix}\right)\right]$$

*Result (1·1) has been arrived at by K. C. Gupta using a different technique. The result is hetherto unpublished.

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{l}\Gamma(b_j - f_j s)\prod_{j=2}^{u}\Gamma(1 - a_j + e_j s)}{\prod_{j=l+1}^{q}\Gamma(1 - b_j + f_j s)\prod_{j=u+1}^{p}\Gamma(a_j - e_j s)}\beta^s\,\Gamma(1 - a_1 + e_1 s)$$

$$\times \frac{d^n}{dx^n}\left( x^{a_1 - 1 - e_1 s}\right) ds$$

$$=(-1)^n x^{a_1 - n - 1} H_{pq}^{lu}\left[\frac{\beta}{x^{e_1}}\middle| \begin{matrix}(a_1 - n, e_1)(a_2, e_2)\dots(a_p, e_p)\\(b_1, f_1)\dots(b_q, f_q)\end{matrix}\right]$$

Similarly

$$\frac{d^n}{dx^n}\left[ x^{a_p - 1} H_{pq}^{lu}\left(\frac{\beta}{x^{e_p}}\middle| \begin{matrix}(a_1, e_1)\dots(a_p, e_p)\\(b_1, f_1)\dots(b_q, f_q)\end{matrix}\right)\right]$$

$$= x^{a_p - n - 1} H_{pq}^{lu}\left[\frac{\beta}{x^{e_p}}\middle| \begin{matrix}(a_1, e_1)\dots(a_{p-1}, e_{p-1}), (a_p - n, e_p)\\(b_1, f_1)\dots(b_q, f_q)\end{matrix}\right] \qquad (1{\cdot}2)$$

$$\frac{d^n}{dx^n}\left[ x^{-b_1} H_{pq}^{lu}\left(\beta x^{f_1}\middle| \begin{matrix}(a_1, e_1)\dots(a_p, e_p)\\(b_1, f_1)\dots(b_q, f_q)\end{matrix}\right)\right]$$

$$= (-1)^n x^{-b_1 - n} H_{pq}^{lu}\left[\beta x^{f_1}\middle| \begin{matrix}(a_1\, e_1)\dots(a_p, e_p)\\(b_1 + n, f_1), (b_2, f_2)\dots(b_q, f_q)\end{matrix}\right] \qquad (1{\cdot}3)$$

$$\frac{d^n}{dx^n}\left[ x^{-b_q} H_{pq}^{lu}\left(\beta x^{f_q}\middle| \begin{matrix}(a_1, e_1)\dots(a_p, e_p)\\(b_1, f_1)\dots(b_q, f_q)\end{matrix}\right)\right]$$

$$= x^{-b_q - n} H_{pq}^{lu}\left[\beta x^{f_q}\middle| \begin{matrix}(a_1, e_1)\dots(a_p, e_p)\\(b_1, f_1)\dots(b_q + n, f_q)\end{matrix}\right] \qquad (1{\cdot}4)$$

In (1·1) to (1·4) the expressions are chosen keeping in view the symmetry in the parameters. The differentiation within the sign of integration can be easily justified, since the $s$-integral on the R. H. S. of (1·0) is absolutely convergent in at least one of the following cases :

(*i*) $\lambda > 0$, $|\arg x| < \frac{1}{2}\pi\lambda$ ; (*ii*) $\lambda \geqslant 0$, $|\arg x| \leqslant \frac{1}{2}\pi\lambda$ and $R(\mu + 1) < 0$.

$\lambda = \sum_{j=1}^{u} e_j - \sum_{j=u+1}^{p} e_j + \sum_{j=1}^{l} f_j - \sum_{j=l+1}^{q} f_j$ and $\mu = \frac{1}{2}(p - q) + \sum_{j=1}^{q}(b_j) - \sum_{j=1}^{p} a_j$

**Particular cases**

(*i*) when $n = 1$, $x = 1/y$ in (1·1) we get

$$-y^2 \frac{d}{dy} H_{pq}^{lu}\left[\beta y^{e_1}\middle| \begin{matrix}(a_1, e_1)\dots(a_p, e_p)\\(b_1, f_1)\dots(b_q, f_q)\end{matrix}\right]$$

$$= H_{pq}^{lu}\left[\beta y^{e_1}\,\middle|\,\begin{matrix}(a_1-1,e_1),(a_2,e_2)\ldots(a_p,e_p)\\(b_1,f_1)\ldots\ldots(b_q,f_q)\end{matrix}\right]$$

$$+(a_1-1)\ H_{pq}^{lu}\left[\beta y^{e_1}\,\middle|\,\begin{matrix}(a_1,e_1)\ldots(a_p,e_p)\\(b_1,f_1)\ldots(b_q,f_q)\end{matrix}\right] \tag{1·5}$$

(*ii*) When $n = 1$, $x = 1/y$ in (1·2) we have

$$-y^2\frac{d}{dy}\ H_{pq}^{lu}\left[\beta y^{e_p}\,\middle|\,\begin{matrix}(a_1,e_1)\ldots(a_p,e_p)\\(b_1,f_1)\ldots(b_q,f_q)\end{matrix}\right]$$

$$=(a_p-1)\ H_{pq}^{lu}\left[\beta y^{e_p}\,\middle|\,\begin{matrix}(a_1,e_1)\ldots(a_p,e_p)\\(b_1,f_1)\ldots(b_q,f_q)\end{matrix}\right]$$

$$-H_{pq}^{lu}\left[\beta y^{e_p}\,\middle|\,\begin{matrix}(a_1,e_1)\ldots(a_{p-1},e_{p-1}),(a_p-1,e_p)\\(b_1,f_1)\ldots(b_q,f_q)\end{matrix}\right] \tag{1·6}$$

(*iii*) When $n = 1$, in (1·3) we get

$$x\frac{d}{dx}\ H_{pq}^{lu}\left[\beta x^{f_1}\,\middle|\,\begin{matrix}(b_1,e_1)\ldots(b_p,e_p)\\(b_1,f_1)\ldots(b_q,f_q)\end{matrix}\right]$$

$$=b_1\,H_{pq}^{lu}\left[\beta x^{f_1}\,\middle|\,\begin{matrix}(a_1,e_1)\ldots(a_p,e_p)\\(b_1,f_1)\ldots(b_q,f_q)\end{matrix}\right]$$

$$-H_{pq}^{lu}\left[\beta x^{f_1}\,\middle|\,\begin{matrix}(a_1,e_1),\ldots(a_p,e_p)\\(b_1+1,f_1)\,(b_2,f_2)\ldots(b_q,f_q)\end{matrix}\right] \tag{1·7}$$

(*iv*) When $n = 1$ in (1·4) we get

$$x\frac{d}{dx}\,H_{pq}^{lu}\left[\beta x^{f_q}\,\middle|\,\begin{matrix}(a_1,e_1)\ldots(a_p,e_p)\\(b_1,f_1)\ldots(b_q,f_q)\end{matrix}\right]$$

$$=b_q\ H_{pq}^{lu}\left[\beta x^{f_q}\,\middle|\,\begin{matrix}(a_1,e_1)\ldots(a_p,e_p)\\(b_1,f_1)\ldots(b_q,f_q)\end{matrix}\right]$$

$$+\ H_{pq}^{la}\left[\beta x^{f_q}\,\middle|\,\begin{matrix}(a_1,e_1)\ldots(a_p,e_p)\\(b_1,f_1)\ldots(b_{q-1},f_{q-1}),(b_q+1,f_q)\end{matrix}\right] \tag{1·8}$$

Now if, in (1·1) to (1·4) $e_j, j = 1, 2, \ldots, u, f_j, j = 1, 2, \ldots 1$ are all equal to $t$, where $t$ is a positive integer then using

$$H_{pq}^{lu}\left[x\,\middle|\,\begin{matrix}(a_1,t),(a_2,t)\ldots(a_p,t)\\(b_1,t),(b_2,t)\ldots(b_q,t)\end{matrix}\right]=\frac{1}{t}\ G_{pq}^{lu}\left[x^{1/t}\,\middle|\,\begin{matrix}a_1,a_2,\ldots a_p\\b_1,b_2,\ldots b_q\end{matrix}\right] \tag{1·9}$$

We obtain similar relations for the Meijer G-function. Further, if $t = 1$, we get from (1·1) to (1·4) the relations given by Bhise[3] equations 2·2, 3·1, 3·2 and 3·3 as particular cases. Also if, in (1·5) to (1·8) *e's* and *f's* are $t$, then with the help of (1·9) we get similar relations for the Meijer G-function. Further, if $t = 1$, we get Bhise[3] equations 5·1, 5·2, 5·3, 5·4 as particular cases. Further specialization of parameters will yield many known results in MacRobert's E-functions. Some more recurrence relations can be easily obtained from (1·5) to (1·8) by elementary addition and subtraction.

**References**


1. Fox, C. The G and H-function as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.* **98** : 395–429, (1961).
2. Gupta, K. C. On the H-function. *Annales de la Societe Scientifique de Bruxelles* T 79, **II**: 97–106, (1965).
3. Bhise V. M. On the derivatives of the Meijer G and MacRobert E-functions. *Proc. Nat. Acad. Sc.* 32(4) : 349–354, (1962).

# Some relation between Hankel and Meijer transforms

*By*

G. K. GOYAL

*Department of Mathematics, University of Rajasthan, Jaipur*

[Received on 1st July, 1966]

**Abstract**

A few relations between Hankel and Meijer transforms are established and some interesting integrals evaluated by their application.

## 1. Introduction

In this paper four theorems involving Hankel and Meijer transforms defined as

$$\Phi(p) = \int_0^\infty (pt)^{\frac{1}{2}} J_\nu(pt) f(t)\, dt,\ R(p) > 0 \tag{1·1}$$

$$\Psi(p) = \sqrt{\tfrac{2}{\pi}}\, p \int_0^\infty (pt)^{\frac{1}{2}} K_\nu(pt) f(t)\, dt,\ R(p) > 0 \tag{1·2}$$

are proved and certain integrals relating to ${}_1F_1$, $S_3$, $K_\nu(x)$, $J_\nu(x)$, $P_n^m(x)$ etc. evaluated.

Throughout this note (1·1) and (1·2) will be symbolically represented as

$$\Phi(p) \underset{v}{\overset{J}{=}} f(t) \text{ and } \Psi(p) \underset{v}{\overset{K}{=}} f(t) \text{ respectively.}$$

If we take $v = \pm \frac{1}{2}$, (1·2) reduce to Laplace transform

$$\Psi(p) = p \int_0^\infty e^{-pt} f(t)\, dt,\ R(p) > 0 \tag{1·3}$$

which will be represented as $\psi(p) \doteqdot f(t)$ hereafter. Further wherever it occurs, the following symbol will represent the series

$$\triangle(\alpha, n) = \alpha/n, (\alpha+1)/n, (\alpha+2)/n, \ldots, (\alpha+n-1)/n.$$

$$\triangle(\alpha\pm\beta, n) = (\alpha+\beta)/n, (\alpha+\beta+1)/n, \ldots, (\alpha+\beta+n-1)/n, (\alpha-\beta)/n, (\alpha-\beta+1)/n, \ldots, (\alpha-\beta+n-1)/n.$$

## 2. Theorem 1

$$\text{If } \Psi(p) \underset{\mu}{\overset{K}{=}} f(t)$$

and $\Phi(p) \underset{v}{\overset{J}{=}} t^\lambda\, \psi(t^{r/s})$, $r$ and $s$ being positive integers

then

$$\left(\frac{p}{2r}\right)^{\lambda+1} (2\pi)^{s-\frac{1}{2}} \left(\frac{r}{4s}\right)^{\frac{1}{2}} \Phi(p) = \int_0^\infty G^{2s,\, r}_{2r,\, 2s}\left\{\left(\frac{2r}{p}\right)^{2r}\left(\frac{t}{2s}\right)^{2s} \middle|\, \begin{matrix} \triangle\left(\frac{\mp\nu-\lambda}{2}, r\right) \\ \triangle\left(\pm\frac{\mu}{2}, s\right) \end{matrix} \right\} \times \frac{f(t)}{t}\, dt, \tag{2·1}$$

provided that the Meijer transform of $|f(t)|$ and the Hankel transform of $|t^{\lambda}\psi(t^{r/s})|$ exist, (2·1) is absolutely convergent and

$$R(\lambda \pm \frac{\mu r}{s} + \nu + \frac{3r}{2s} + \tfrac{3}{2}) > 0 ; R(p) > 0.$$

**Proof**

Substitute the value of $\Psi(x^{r/s})$ from (1·2) in

$$\Phi(p) = \int_0^\infty (px)^{\frac{1}{2}} J_\nu(px)\, x^\lambda\, \psi(x^{r/s})\, dx,$$

we get

$$\Phi(p) = \int_0^\infty (px)^{\frac{1}{2}} J_\nu(px)\, x^\lambda\, \{\sqrt{\tfrac{2}{\pi}}\, x^{r/s} \int_0^\infty p^{r/2s}\, t^{\frac{1}{2}}\, K_\mu(tx^{r/s}) f(t)\, dt\}\, dx \qquad (A)$$

Changing the order of integration we obtain

$$\phi(p) = \sqrt{\frac{p}{2\pi}} \int_0^\infty t^{\frac{1}{2}} f(t)\, dt \int_0^\infty x^{\lambda + \frac{3r}{2s} + \frac{1}{2}}\, G^{1\,0}_{0\,2}\left\{\frac{p^2 x^2}{4} \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2}\right\} G^{2\,0}_{0\,2}\left\{\frac{t^2 x^{2r/s}}{4} \,\middle|\, \frac{\mu}{2}, -\frac{\mu}{2}\right\} dx$$

as (3, p. 219)

$$J_\nu(x) = G^{1\,0}_{0\,2}(x^2/2 \mid \frac{\nu}{2}, -\frac{\nu}{2}) \text{ and } K_\mu(x) = \tfrac{1}{2}\, G^{2\,0}_{0\,2}(x^2/4 \mid \frac{\mu}{2}, -\frac{\mu}{2})$$

Now integrating the $x$-integral with the help of the result given by Saxena[4], we arrive at the required result after a little simplification on using (3, p. 209)

$$x^\sigma G^{m\,n}_{p\,q}\left\{x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right\} = G^{m\,n}_{p\,q}\left\{x \,\middle|\, \begin{matrix} a_r+\sigma \\ b_s+\sigma \end{matrix}\right\} \qquad (2·2)$$

The change of the order of integration above is justified by de la Roussin's theorem as we observe that

($i$) the $t$-integral is convergent as $f(t)$ is positive and tends to zero and

$$R(\tfrac{3}{2} \pm \frac{\mu r}{s}) > 0,$$

($ii$) the $x$-integral is convergent if $R(\lambda + \nu + \frac{3r}{2s} \pm \frac{\mu r}{s} + \frac{3}{2}) > 0$ as $K_\mu(x) = 0\,(ax^\mu + bx^{-\mu})$ for small $x$ and vanishes exponentially for large $x$ and $J_\nu(x) = 0(x^\nu)$ for small $x$

$= 0\{x^{-\frac{1}{2}} \cos(x+a)\}$ for large $x$

**Corollary.** If $\psi(p) \doteqdot f(t)$

and $\Phi(p) \underset{v}{\overset{J}{=}} t^{2m-n-7/2}\, \Psi(t^2)$

then

$$\frac{2^{\nu+1}\,\Gamma(\nu+1)}{\Gamma m\, p^{\nu+\frac{1}{2}}}\, \Phi(p) = \int_0^\infty {}_1F_1(m\,;\,\nu+1\,;\,-p^2/4t)\, f(t)\, dt, \qquad (2·3)$$

where $R(2m-n+\nu) > 0$, $R(2m-n-\frac{1}{2}) < 0$, $R(p) > 0$.

This is easily obtained on putting $r=2$, $s=1$, $\mu = \pm\frac{1}{2}$, $\lambda = 2m-n-\frac{7}{2}$, in the above theorem.

**Example**

Let $f(t) = t^{\mu-1} e^{-at}$, then substituting the value of $\psi(p)$ from (1, p. 144) and $\phi(p)$ from (2, p. 24) in (2·3)

we get

$$\int_0^\infty t^{\mu-m-1} e^{-at} {}_1F_1(m\,;\,\nu+1\,;\,-p^2/4t)\,dt = \frac{\Gamma\mu\,\Gamma(1+\nu)}{\Gamma m}\left[\frac{\Gamma(m+\frac{\nu-n}{2})\,\Gamma(\mu-m+\frac{n-\nu}{2}+2)}{a^{2\mu-2m+n-\nu+4}\,\Gamma(2+\mu)\,\Gamma(1+\nu)}\right.$$

$$\times\, {}_1F_2\left\{\begin{matrix} m+\frac{\nu-n}{2} \\ m+\frac{\nu-n}{2}-\mu-1,\ \nu+1\end{matrix}\,;\,\frac{ap^2}{4}\right\} + \frac{p^{2\mu-2m+n-\nu+4}\,\Gamma(m+\frac{\nu-n}{2}-\mu-2)}{2^{2\mu-2m+n-\nu+4}\,\Gamma(\mu-m+\frac{\nu+n}{2}+3)}$$

$$\left.\times\, {}_1F_2\left\{\begin{matrix} \mu+2 \\ \mu-m\pm\frac{\nu}{2}+\frac{n}{2}+3\end{matrix}\,;\,\frac{ap^2}{4}\right\}\right] \tag{2·4}$$

where $R(\mu) > 2R(m)$, $R(a) > 0$, $R(p) > 0$.

### 3. Theorem 2.

If
$$\psi(p) \underset{\mu}{\overset{k}{=}} f(t),\ R(p) > 0.$$

and
$$\Phi(p) \underset{\nu}{\overset{J}{=}} t^\lambda\,\psi(t^{-r/s}),\ r \text{ and } s \text{ being positive integers}$$

then

$$\sqrt{\frac{r}{2\pi s}}\,\frac{(2\pi)^s}{2s}\left(\frac{p}{2r}\right)^{\lambda+1}\Phi(p) = \int_0^\infty G^{2s+r,\,0}_{0,\,2s+2r}\left\{\left(\frac{p}{2r}\right)^{2r}\left(\frac{t}{2s}\right)^{2s}\middle|\,\Delta(\tfrac{3}{4}\pm\tfrac{\mu}{2},\,s),\right.$$

$$\left.\Delta\left(\frac{\lambda\pm\nu}{2}+\tfrac{3}{4},\,r\right)\right\}\frac{f(t)}{t}\,dt, \tag{3·1}$$

provided that the Meijer transform of $|f(t)|$ and the Hankel transform of $|t^\lambda\,\psi(t^{-r/s})|$ exist, (3·1) is absolutely convergent and $R(\lambda\pm\frac{\mu r}{s}-\frac{3r}{2s}+1) < 0$.

**Proof**

Using (1·2) to get the value of $\psi(t^{-r/s})$ and then substituting it in

$$\Phi(p) = \int_0^\infty (px)^{\frac{1}{2}}\,J_\nu(px)\,x^\lambda\,\psi(x^{-r/s})\,dx, \tag{B}$$

we get, on changing the order of integration in the above,

$$\Phi(p) = \sqrt{\frac{p}{2\pi}}\int_0^\infty t^{\frac{1}{2}} f(t)\,dt\int_0^\infty x^{\lambda-\frac{3r}{2s}+\frac{1}{2}}\,J_\nu(px)\,K_\mu(tx^{-r/s})\,dx,$$

$$= \sqrt{\frac{p}{2\pi}}\int_0^\infty t^{\frac{1}{2}} f(t)\,dt\int_0^\infty x^{\lambda-\frac{3r}{2s}+\frac{1}{2}}\,G^{1\,0}_{0\,2}\left(p^2x^2/4\,\middle|\,\tfrac{\nu}{2},-\tfrac{\nu}{2}\right)G^{2\,0}_{0\,2}\left(\frac{t^2x^{-2r/s}}{4}\,\middle|\,\tfrac{\mu}{2},-\tfrac{\mu}{2}\right)dx$$

Now integrating the $x$–integral by Saxena's result (3, p. 209) we obtain (3·1) on using

$$G^{m\,n}_{p\,q}\left(1/x\,\middle|\,\begin{matrix}a_r\\b_s\end{matrix}\right) = G^{n\,m}_{q\,p}\left(x\,\middle|\,\begin{matrix}1-b_s\\1-a_r\end{matrix}\right). \tag{3·2}$$

The change of the order of integration in $(B)$ is justified as the investigations show that

(i) the $t$-integral is convergent as $f(t)$ is positive and tends to zero and $R(\frac{3}{2}\pm\frac{\mu r}{s}) > 0$.

(ii) the $x$-integral is convergent if $R(\lambda\pm\frac{\mu r}{s}-\frac{3r}{2s}+1) < 0$ as $K_\mu(tx^{-r/s}) = 0\,(a\,x^{-\mu r/s} + b\,x^{\mu r/s})$ for large $x$ and vanishes exponentially for small $x$ and $J_\nu(x) = 0(x^\nu)$ for small $x$ and $0\,(x^{-\frac{1}{2}}\cos\overline{x+a})$ for large $x$.

**Corollary**

If
$$\psi(p)\ \frac{k}{\overline{\mu}}\ f(t)$$
and
$$\phi(p)\ \frac{J}{\overline{\nu}}\ t^{-\sigma-\frac{1}{2}}\ \Psi(1/t)$$
then
$$\frac{2^{\sigma-3}}{p^{\sigma-3/2}}\sqrt{\pi}\ \Phi(p) = \int_0^\infty S_3\left\{\frac{2\mu+3}{4}, \frac{-2\mu+3}{4}, \frac{\nu-\sigma+1}{2}, \frac{1-\nu-\sigma}{2}, pt/4\right\} f(t)\frac{dt}{t^2}, \quad (3\cdot3)$$
where $R(\nu-\sigma) > 0$, $R(\sigma\pm\mu+1) > 0$ and $S_n(a, b, c, d\,; x^{\frac{1}{2}}) = x^{\frac{1}{2}}$
$$\times\ G^{n\ 0}_{0\ 4}\left[x \mid a, b, c, d\right],\ n=1, 2, 3, 4.$$
This is easily obtained on taking $r=s=1$ and $\lambda=-\sigma-\frac{1}{2}$ in the above theorem.

**Example**

Let $\quad f(t) = t^{\frac{1}{2}-\mu}\,[I_{2\mu}(a\sqrt{t})\,J_{-2\mu}(a\sqrt{t}) - J_{2\mu}(a\sqrt{t})\,I_{-2\mu}(a\sqrt{t})]$

then substituting $\Psi(p)$ from (2, p. 149) and $\Phi(p)$ from (2, p. 48) in (3·3) we have
$$\int_0^\infty t^{-\mu-3/2}\,[I_{2\mu}(a\sqrt{t})\,J_{-2\mu}(a\sqrt{t}) - J_{2\mu}(a\sqrt{t})\,I_{-2\mu}(a\sqrt{t})]$$
$$\times\ S_3\left\{\frac{2\mu+3}{4}, \frac{-2\mu+3}{4}, \frac{\nu-\sigma+1}{2}, \frac{1-\nu-\sigma}{2}; \frac{pt}{2}\right\} dt$$
$$= -\frac{\pi\,p^{3-\sigma+\nu}\,\Gamma\frac{1}{2}(\nu-\sigma-\mu+\frac{7}{2})\,\Gamma\frac{1}{2}(\sigma+3\mu-\nu-\frac{1}{2})}{2^{\sigma+3\mu-\nu-2}\,a^{2\nu-2\sigma-2\mu+5}\,\Gamma(1+\nu)}$$
$$\times\ {}_2F_1\left\{\begin{matrix}\frac{1}{2}(\nu-\sigma-\mu+\frac{7}{2}), \frac{1}{2}(\nu-\sigma-3\mu+\frac{5}{2})\\ 1+\nu\end{matrix}; \frac{4p^2}{a^4}\right\} \quad (3\cdot4)$$
where $R(p) > 0$, $a > 0$, $R(\mu) > 1$, $R\left(\frac{\mu}{2}+\frac{\sigma}{4}\right) > 1$.

## 4. Theorem 3

If
$$\Phi(p)\,\frac{J}{\overline{\mu}}\,f(t),\ R(p) > 0$$
and
$$\Psi(p)\ \frac{k}{\overline{\nu}}\ t^\lambda\ \Phi\,(t^{-r/s}),\ r \text{ and } s \text{ being positive integers}$$
then
$$\frac{p^\lambda(2\pi)^{r-\frac{1}{2}}}{(2r)^{\lambda+\frac{1}{2}}(2s)^{\frac{1}{2}}}\ \psi(p) = \int_0^\infty G^{s+2r,\,0}_{0,2s+2r}\left\{\left(\frac{p}{2r}\right)^{2r}\left(\frac{t}{2s}\right)^{2s} \middle|\ \triangle(\tfrac{1}{4}+\tfrac{\mu}{2}, s),\right.$$
$$\left.\triangle(\tfrac{3}{4}\pm\tfrac{\nu}{2}+\tfrac{\lambda}{2}, r), \triangle(\tfrac{1}{4}-\tfrac{\mu}{2}, s)\right\} f(t)\,dt, \quad (4\cdot1)$$

provided that the Hankel transform of $|f(t)|$ and the Meijer transform of $|t^{\lambda}\,\phi(t^{-r/s})|$ exist, (4·1) is absolutely convergent and $R(\lambda-\frac{r}{2s}-\frac{\mu r}{s}+\frac{3}{2}\pm\nu)<0$.

**Proof**

Obtain the value of $\phi(t^{-r/s})$ from (1·1) and substitute it

in
$$\psi(p)=\sqrt{2/\pi}\;p\int_0^{\infty}(px)^{\frac{1}{2}}\,K_\nu(px)\,x^{\lambda}\,\phi(x^{-r/s})\,dx,$$

we get
$$\psi(p)=\sqrt{\frac{2}{\pi}}\;p\int_0^{\infty}(px)^{\frac{1}{2}}\,K_\nu(px)\,x^{\lambda}\left\{\int_0^{\infty}x^{-r/2s}\,t\,J_\mu(t\,x^{-r/s})\,f(t)\,dt\right\}dx \qquad (C)$$

Changing the order of integration we obtain

$$\psi(p)=\left(\frac{2p^3}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}t^{\frac{1}{2}}\,f(t)\,dt\int_0^{\infty}x^{\lambda-r/2s+\frac{1}{2}}\;G^{2\,0}_{0\,2}\left\{\frac{p^2x^2}{4}\,\middle|\,\frac{\nu}{2},-\frac{\nu}{2}\right\}$$
$$\times\;G^{0\,1}_{2\,0}\left\{\frac{x^{2r/s}}{t^2}\,\middle|\,1-\frac{\mu}{2},\;1+\frac{\mu}{2}\right\}dx$$

Now integrating the $x$-integral by Saxena's result we arrive at (4·1).

The change of the order of integration in $(C)$ is justified by de la Valee Poussin's theorem *viz.*

(*i*) $t$–integral is convergent as $f(t)$ is positive and tends to zero and $R(-\frac{r}{2s}-\frac{\nu r}{s}+1)>0$.

(*ii*) $x$–integral converges as $R(\lambda-\frac{r}{2s}-\frac{\mu r}{s}\pm\nu+\frac{3}{2})<0$.

**Corollary**

If
$$\phi(p)\,\frac{J}{\mu}\,f(t),\; R(p)>0$$

and
$$\Psi(p)\doteqdot t^{-\frac{1}{2}}\,\phi(1/t)$$

then
$$\sqrt{\frac{2}{p^2}}\;\psi(p)=\int_0^{\infty}t^{\frac{1}{2}}\,J_\mu(\sqrt{2pt})\,K_\mu(\sqrt{2pt})\,f(t)\,dt, \qquad (4·2)$$

where the Hankel transform of $|f(t)|$ and the Laplace transform of $|t^{-\frac{1}{2}}\,\phi(1/t)|$ exist, (4·2) is absolutely convergent and $R(\mu)>0$.

This is obtained on putting $r=s=1$, $\nu=\pm\frac{1}{2}$ and $\lambda=-\frac{1}{2}$ in the theorem.

**Example**

Take
$$f(t)=t^{\frac{1}{2}}(a^2+t^2)^{\frac{\nu}{2}-1}\,P^{-\mu}_{1-\nu}\left[a(a^2+t^2)^{-\frac{1}{2}}\right]$$

then putting the values of $\psi(p)$ from (2, p. 45) and $\Psi(p)$ from (1, p. 146) in (4·2) we get

$$\int_0^{\infty}t(a^2+t^2)^{\frac{\mu}{2}-1}\,P^{-\mu}_{1-\nu}\left[a(a^2+t^2)^{-\frac{1}{2}}\right]J_\mu(\sqrt{2pt})\,K_\mu(\sqrt{2pt})\,dt=\frac{2^{3/2}\,(ap)^{\nu/2}}{\Gamma(2-\nu+\mu)}\,K_\nu(2\sqrt{ap}) \qquad (4·3)$$

where $\qquad 0 < R(\mu) < \frac{1}{2},\ R(p) > 0,\ R(\nu) < \frac{3}{2}.$

**5. Theorem 4**

If $\qquad \Phi(p) \underset{\mu}{\overset{J}{=}} f(t),\ R(p) > 0$

and

$$\Psi(p) \underset{\nu}{\overset{k}{=}} t^{\lambda}\ \Phi(t^{r/s}),\ r \text{ and } s \text{ being positive integers then}$$

$$\frac{p^{\lambda}\,(2\pi)^{r-\frac{1}{2}}}{(2r)^{\lambda+\frac{1}{2}}\,(2s)^{\frac{1}{2}}}\,\psi(p) = \int_0^{\infty} G^{s\ 2r}_{2r\ 2s}\left\{\left(\frac{2r}{p}\right)^{2r}\left(\frac{t}{2s}\right)^{2s}\middle|\begin{matrix}\triangle(\mp\nu/2-\lambda/2,\ r)\\ \triangle(\pm\mu/2,\ s)\end{matrix}\right\} f(t)\,dt, \tag{5·1}$$

provided that the Hankel transform of $|f(t)|$ and Meijer transform of $|t^{\lambda}\ \phi(t^{r/s})|$ exist, (5·1) is absolutely convergent and $R(\lambda) < -1$,

$$R\left(\lambda + \frac{r}{2s} + \frac{\mu r}{s} \pm \nu + \frac{3}{2}\right) > 0.$$

**Proof**

Substitute the value of $\Phi(t^{r/s})$ from (1·1) in

$$\psi(p) = \sqrt{\frac{2}{\pi}}\ p\int_0^{\infty} p(x)^{\frac{1}{2}}\ K_{\nu}(px)\ x^{\lambda}\ \ \Phi(x^{r/s})\ dx,$$

we get on changing the order of integration in the above

$$\psi(p) = \left(\frac{2p^3}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty} t^{\frac{1}{2}}\,f(t)\,dt\int_0^{\infty} x^{\lambda+\frac{r}{2s}+\frac{1}{2}}\ J_{\mu}(t\,x^{r/s})\ K_{\nu}(px)\,dx \tag{D}$$

Now integrating the $x$-integral by Saxena's result we arrive at (5·1) after a little simplification.

The change of the order of integration is justified as

(*i*) The convergence of $t$-integral implies that $R\left(\frac{r}{2s} + \frac{\mu r}{s} + 1\right) > 0$ and $f(t)$ to be positive and tending to zero,

(*ii*) $x$-integral is convergent as $R\left(\lambda + \frac{r}{2s} + \frac{\mu r}{s} \pm \nu + \frac{3}{2}\right) < 0.$

**Corollary**

If $\qquad \Phi(p) \underset{\mu}{\overset{J}{=}} f(t)$

and $\qquad \psi(p) \overset{\nu}{\doteqdot} t^{\frac{\nu}{2}-\frac{3}{4}}\ \Phi(\sqrt{t})$

then

$$\frac{\Gamma(1+\mu)\ 2^{\mu}\ p^{\frac{1}{2}(\mu+\nu-1)}}{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}\ \psi(p) = \int_0^{\infty} t^{\mu+\frac{1}{2}}\ {}_1F_1\left\{\frac{\mu+\nu+1}{2}\ ;\ \mu+1\ ;\ \frac{-t^2}{4p}\right\} f(t)\,dt, \tag{5·2}$$

where $R(p) > 0,\ R(\mu+1) < -2R(\nu) < \frac{1}{2}$ and (5·2) is absolutely convergent.

This is readily obtained on taking $\nu = \pm\frac{1}{2},\ \lambda = \nu = \frac{3}{4},\ r = 1,\ s = 2$ in the above theorem.

**Example**

$$\text{Let} \qquad f(t) = 2^{-\frac{1+\mu}{2}} t^{\frac{2\mu+1}{2}} e^{-t^2/2} {}_1F_1\left(\frac{1+\mu}{2} - k\,;\, 1+\mu\,;\, t^2/2\right)$$

then substituting the values of $\Phi(p)$ from (2, p. 83) and $\Psi(p)$ from (1, p. 144) we obtain

$$\int_0^\infty t^{2\mu+1} e^{-t^2/2} {}_1F_1\left(\frac{1+\mu+\nu}{2}\,;\, 1+\mu\,;\, \frac{-t^2}{4p}\right) {}_1F_1\left(\frac{1+\mu}{2} - k\,;\, 1+\mu\,;\, t^2/2\right) dt$$

$$= \frac{\{\Gamma(\mu+1)\}^2\, \Gamma(\frac{\nu}{2}+k)\, 2^{\frac{1}{2}(3\mu+\nu+1)}\, p^{\frac{1}{2}(\mu+\nu-1)}}{\Gamma\left(k+\frac{1+\mu}{2}\right) \Gamma\frac{1}{2}(1+\mu+\nu)\, (2p+1)^{k+\nu/2}} \qquad (5\cdot3)$$

where $R(\mu+1) > 0$, $R(p) > 0$, $R(k+\frac{1}{2}+\frac{\nu}{2}) > 0$.

**Acknowledgement**

The author is grateful to Dr. K. C. Sharma of the University of Rajasthan, Jaipur for his guidance in the preparation of this paper.

**References**

1. Erdelyi, A. *et al.*, Tables of integral transforms, Vol I, (1954).
2. Erdelyi, A. *et al.*, Tables of integral transforms, Vol. II, (1954)
3. Erdelyi, A. *et al.*, Higher Transcendental functions. Vol. I, (1954).
4. Saxena, R. K. Some theorems on generalised Laplace Transforms—I. *Proc. Nat. Inst. Sci., India*, 28 A (4) : (1960).

# Some Integral Equations and Integrals

*By*

K. C. RUSIA

*Department of Applied Mathematics, Government Engineering College, Jabalpur*

[Received on 2nd July, 1966]

**Abstract**

Some Integral equations are solved and certain integrals are evaluated by the application of Laplace Transform.

**Introduction**

**1.** Widder[1] obtained the solution of an integral equation involving Laguerre polynomial by the application of Laplace transform. Using the same method, Bharatiya[2,3] and Khandekar[4] have obtained the solutions of the integral equations involving Bessel function, generalized Bateman function and generalized Laguerre polynomial respectively.

In the present paper we obtain the solutions of the integral equations involving Bessel functions, Theta functions and Bateman function respectively. We also evaluate certain integrals. The method adopted is the same as that of Widder[1].

2. Results required in the proof:

(2·1) We shall represent Laplace transform

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt,\ Re\, p > 0, \text{ by } \bar{f}(p) = L\{f(t)\}.$$

We have from [5, p. 129, p. 131, p. 185, p. 197, p. 195, p. 257, p. 214, p. 162, p. 171, p. 144].

(2·2) If $f(0) = f'(0) = \ldots = f^{n-1}(0) = 0$, then $L\{f^n(t)\} = p^n \bar{f}(p)$

$$(2\cdot3) \qquad L\left\{\int_0^t f_1(u) f_2(t-u)\, du\right\} = \bar{f}_1(p)\, \bar{f}_2(p)$$

$$(2\cdot4) \qquad L\left\{ t^{\frac{n}{2}} J_n(2a^{\frac12} t^{\frac12})\right\} = a^{\frac{n}{2}} p^{-n-1} e^{-\frac{a}{p}},\ Re\, n > -1$$

$$(2\cdot5) \qquad L\left\{ t^{\frac{n}{2}} I_n(2a^{\frac12} t^{\frac12})\right\} = a^{\frac{n}{2}} p^{-n-1} e^{\frac{a}{p}},\ Re\, n > -1$$

$$(2\cdot6) \qquad L\{t^n I_n(at)\} = 2^n \pi^{-\frac12} \Gamma(n+\tfrac12)\, a^n (p^2-a^2)^{-n-\frac12}\ Re\, n > -\tfrac12,\ Re\, p > |Re\, a|$$

$$(2\cdot7) \qquad L\{\theta_2(0/i\pi t)\} = p^{-\frac12} \tanh(p^{\frac12})$$

$$(2\cdot8) \qquad L\{\theta_3(0/i\pi t)\} = p^{-\frac12} \operatorname{ctnh}(p^{\frac12})$$

$$(2\cdot9) \qquad L\left\{ -\left[\frac{\partial}{\partial v}\theta_1\left(\frac{v}{2}/i\pi t\right)\right]_{v=0}\right\} = \operatorname{sech}(p^{\frac12})$$

(2·10) $L\{\,\hat{\theta}_2\,(\tfrac{1}{2}/i\,\pi\,t)\,\} = p^{-!}\ \text{sech}\ (p^{\frac{1}{2}})$

(2·11) $L\left\{-\left[\frac{\partial}{\partial v}\,\hat{\theta}_4\cdot\left(\frac{v}{2}\,/i\,\pi\,t\right)\right]_{v=0}\right\} = \text{c sch}\ (p^{\cdot})$

(2·12) $L\{\,\theta_4\,(0/i\,\pi\,t\ \} = p^{-\frac{1}{2}}\ \text{c sch}\ (p^{\frac{1}{2}})$

(2·13) $L\{\,K_{2n+2}\,(t)\,\} = \dfrac{2\,(1-p)^n}{(1+p)^{n+2}},\ Re\,p > -1$

(2·14) $L\{\,\sinh\,(a\,t)\,\} = \dfrac{a}{p^2-a^2},\ Re\,p > |\,Re\,a\,|$

(2·15) $L\{\,P_{2n}\,(\cosh t)\,\} = \dfrac{(p^2-1^2)\,(p^2-3^2)\ldots[p^2-(2n-1)^2]}{p(^2-2^2)\,(p^2-4^2)\ldots[p^2-(2n)^2]},\ Re\,p > 2n$

(2·16) $L\{P_{2n+1}\,(\cosh t)\,\} = \dfrac{p\,(p^2-2^2)\,(p^2-4^2)\ldots.[p^2-(2n)^2]}{p^2-1^2)\,(p^2-3^2)\ldots.[p^2-(2n+1)^2]},$

$Re\,p > 2n+1$

(2·17) $L\{\,t^{n-1}\,e^{-\alpha t}\,\} = \dfrac{\Gamma(n)}{(p+\alpha)^n},\ Re\,n > 0,\ Re\,p > -Re\,\alpha$

**3. Theorem I**

The integral equation

(3·1) $$\int_0^t (t-u)^{n/2}\ I_n\,[2a^{\frac{1}{2}}\,(t-u)^{\frac{1}{2}}]\,g\,(u)\,du = f\,(t)$$

has the solution

(3·2) $$g\,(t) = a^{-n/2}\int_0^t J_0\,[2a^{\frac{1}{2}}\,(t-u)^{\frac{1}{2}}]\,f^{n+2}\,(u)\,du$$

provided that $f\,(t)\ \varepsilon\ c^{n+2},\ 0 \leqslant t < \infty$ and $f\,(0) = f'\,(0) = \ldots. = f^{n+1}\,(0) = 0.$

**Proof**

Applying (2·3) to (3·1), using (2·5) and rearranging the terms we get

(3·3) $\bar{g}\,(p) = a^{-n/2}\,[p^{-1}\,e^{-\alpha/p}]\,[p^{n+2}\,\bar{f}\,(p)].$

Applying (2·4), (2·2) and (2·3) to (3·3) we get (3·2).

**Theorem II**

The integral equation

(3·4) $$\int_0^t (t-u)^n\,I_n\,[a\,(t-u)]\,g\,(u)\,du = f\,(t)$$ has the solution

(3·5) $$g\,(t) = \frac{2^{-n}\,\pi\,a^{-n}}{\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})}\int_0^t I_0\,[a\,(t-u)]\,[(D^2-a^2)^{n+1}f\,(u)]\,du$$

provided that $f\,(t)\ \varepsilon\ c^{2n+2},\ 0 \leqslant t < \infty$ and $f\,(0) = f'\,(0) = \ldots. = f^{2n+1}\,(0) = 0.$

**Proof**

Applying (2·3) to (3·4) using (2·6) and rearranging the terms we get

$$(3{\cdot}6) \qquad \bar{g}\,(p) = \frac{2^{-n}\,\pi\, a^{-n}}{\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})}\,\left[\pi^{-\frac{1}{2}}\,\Gamma(\tfrac{1}{2})\,(p^2-a^2)^{-\frac{1}{2}}\right]\left[(p^2-a^2)^{n+1}\,\bar{f}\,(p)\right].$$

Applying (2·6), (2·2) and (2·3) to (3·6), we get (3·5).

## 4. Theorem III

The integral equation

$$(4{\cdot}1) \qquad \int_0^t \theta_2\,(0/i\,\pi\,(t-u)\,)\;g\,(u)\,du = f\,(t) \text{ has the solution}$$

$$(4{\cdot}2) \qquad g\,(t) = \int_0^t \theta_3\,(0/i\,\pi\,(t-u)\,)\,f^1\,(u)\,du$$

provided that $\quad f\,(t)\ \varepsilon\ c',\ 0 \leqslant t < \infty$ and $f\,(0) = 0$.

**Proof**

Applying (2·3) to (4·1), using (2·7) and rearranging the terms we get

$$(4{\cdot}3) \qquad \bar{g}\,(p) = [p^{-\frac{1}{2}}\,\text{ctnh}\,(p^{\frac{1}{2}})\,]\,[p\,\bar{f}\,(p)].$$

Applying (2·8), (2·2) and (2·3) to (4·3), we get (4·2).

**Theorem IV**

The integral equation

$$(4{\cdot}4) \qquad \int_0^t k_{2n+2}\,(t-u)\;g\,(u)\,du = f\,(t) \quad \text{has the solution}$$

$$(4{\cdot}5) \qquad g\,(t) = \frac{(-1)^n}{2\,(n-1)\,!}\int_0^t e^{t-u}\,(t-u)^{n-1}\;\left[(D+1)^{n+2} f\,(u)\right]du$$

provided that $f\,(t)\ \varepsilon\ c^{n+2},\ 0 \leqslant t < \infty$ and $f\,(0) = f'\,(0) = \ldots\ldots = f^{n+1}\,(0) =$ .

**Proof**

Applying (2·3) to (4·4), using (2·13) and rearranging the terms we get

$$(4{\cdot}6) \qquad \bar{g}\,(p) = \frac{(-1)^n}{2\,(\,n\ \ 1)\,!}\left[\frac{(n-1)\,!}{(p-1)^n}\right]\left[\,(p+1)^{n+2}\,\bar{f}\,(p)\right]$$

Applying (2·17), (2·2) and (2·3) to (4·6) we get (4·5).

It should be noted that the above theorem can not be derived from the theorems of[3].

**5.** In this section we evaluate some integrals.

$$(5{\cdot}1) \qquad \int_0^t P_{2n}\,[\cosh\,(t-u)\,]\;P_{2n+1}\,(\cosh\,u)\,du$$

$$= \int_0^t P_{2n+1}\,[\cosh\,(t-u)\,]\;P_{2n}\,(\cosh\,u)\,du = \frac{\sinh\,[(2n+1)\,t]}{2n+1}$$

This is proved by using (2·3), (2·14), (2·15) and (2·16).

(5·2)
$$\int_0^t (t-u)^{n/2} J_n\,(2a^{\frac{1}{2}}\,(t-u)^{\frac{1}{2}}\,).\ u^{n/2}\, I_n\,(2a^{\frac{1}{2}}\,u^{\frac{1}{2}})\,du$$

$$= \int_0^t (t-u)^{n/2}\, I_n\,(2a^{\frac{1}{2}}\,(t-u)^{\frac{1}{2}}).\ u^{n/2} J_n\,(2a^{\frac{1}{2}}\,u^{\frac{1}{2}})\,du \;=\; \frac{a^n\, t^{2n+1}}{(2n+1)\,!}$$

This is proved by using (2·3), (2·4), (2·5) and (2·17).

(5·3)
$$\int_0^t \theta_2\,(0/i\,\pi\,(t-u)\,)\left[-\frac{\partial}{\partial v}\ \hat{\theta}_4\ \ (v/2/i\,\pi\,u)\right]_{v=0} du$$

$$= \int_0^t \theta_2\,(0/i\,\pi\,u)\left[-\frac{\partial}{\partial v}\ \hat{\theta}_4\,(v/2/i\,\pi\,(t-u)\,)\right]_{v=0} du$$

$$= \hat{\theta}_2\,(\tfrac{1}{2}/i\,\pi\,t).$$

This is proved by using (2·3), (2·7), (2·11) and (2·10).

(5·4)
$$\int_0^t \theta_3\,(0/i\,\pi\,(t-u)\,)\left[-\frac{\partial}{\partial v}\ \ \theta_1\,(v/2/i\,\pi\,u)\right]_{v=0} du$$

$$= \int_0^t \theta_3\,(0/i\,\pi\,u)\left[-\frac{\partial}{\partial v}\ \theta_1\,(v/2/i\,\pi\,(t-u))\right]_{v=0} du$$

$$= \theta_4\,(0/i\,\pi\,t).$$

This is proved by using (2·3), (2·8), (2·9) and (2·12).

In conclusion I wish to express my grateful thanks to Professor B. R. Bhonsle for his valuable guidance in the preparation of the paper.

I am also very much thankful to the referee for valuable suggestions.

**References**

1. Widder, D. V. The inversion of a convolution transform whose kernel is a Laguerre polynomial. *Amer. Math. Month.* 70 (3) : 291–293, (1963).

2. Bharatiya, P. L. The inversion of a convolution transform whose kernel is a Bessel function. *Amer. Math. Month.* 72 (4) : (1965).

3. Bharatiya, P. L. The inversion of a convolution transform whose kernel is a generalized Bateman's function. *J. Indian Math. Soc., New Series,* 28 (3 and 4) : (1964).

4. Khandeker, P. R. On the convolution transform involving generalized Laguerre polynomial as its kernel. *J. De Mathematiques, Pures Et Appliquees, Paris,* 44 : 195–197, (1965).

5. Erdelyi, A. Tables of integral transforms. *McGraw Hill, New York,* Vol. I : (1954).

# Integration of Some E-functions with respect to their Parameters

*By*

S. D. BAJPAI

*Department of Mathematics, Shri G. S. Technological Institute, Indore*

[Received on 27th July, 1966]

**Abstract**

In this paper we have evaluated some integrals of E-functions with respect to their parameters, and employed them to sum certain series of products of two E-functions.

## 1. Introduction

Mac-Robert[5] and Ragab[6, 7, 8] have obtained a number of integrals of E-functions with respect to their parameters. The object of this paper is to establish some further generalized integrals of this type, and employ them to sum certain series of products of two E-functions. In what follows $m$ is a positive integer and $\Delta(m, \alpha)$ represents the set of parameters

$$\frac{\alpha}{m}, \frac{\alpha+1}{m}, \ldots\ldots, \frac{\alpha+m-1}{m}.$$

The following formulae are required in the proofs.

If $Re\,\gamma > Re\,\beta > 0$,

$$(1\cdot1)\quad \int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1} E\left(\begin{matrix}\alpha_1, \ldots, \alpha_p : \\ \rho_1, \ldots, \rho_q\end{matrix} x^{-m} z\right) dx$$

$$= \Gamma(\gamma-\beta)\, m^{\beta-\gamma} E\left(\begin{matrix}\alpha_1, \ldots, \alpha_p, \Delta(m, \beta) \\ \rho_1, \ldots, \rho_q, \Delta(m, \gamma)\end{matrix} : z\right),$$

which follows from [3, p. 414, (2)].

If $Re\,\beta > 0$,

$$(1\cdot2)\quad \int_0^\infty x^{\beta-1} e^{-x} E\left(\begin{matrix}\alpha_1, \ldots, \alpha_p \\ \rho_1, \ldots, \rho_q,\end{matrix} : x^{-m}z\right) dx$$

$$= (2\pi)^{\frac{1}{2}-\frac{1}{2}m}\, m^{\beta-\frac{1}{2}} E\left(\begin{matrix}\alpha_1, \ldots, \alpha_p, \Delta(m, \beta) : m^{-m}z \\ \rho_1, \ldots, \rho_q\end{matrix}\right),$$

which follows from [3, p. 415, (5)].

**2.** (*i*) *The first integral.* If $p \geqslant q$, $|\,\text{amp.}\, z\,| < \frac{1}{2}(p-q)\pi$, $Re\,(\alpha_r - \rho_r) > 0$, $(r = 1, 2, \ldots, q)$, $Re\,(\alpha_r) > 0$, $(r = 1, 2, \ldots, p)$, $m$ is a positive integer, $\alpha$ is such that E-functions exist,

$$(2\cdot1)\quad \frac{1}{2\pi i}\int \frac{\Gamma(\zeta)}{\Gamma(\alpha-\zeta)} E\left(\begin{matrix}\Delta(m, \alpha_1 - m\zeta), \ldots, \Delta(m, \alpha_p - m\zeta) \\ \alpha, \Delta(m, \rho_1 - m\zeta), \ldots, \Delta(m, \rho_q - m\zeta)\end{matrix} : -z\, m^{-m(p-q)}\right)$$

$$\times \{z\, m^{-m(p-q)}\}^{\zeta}\, d\zeta$$

$$= \frac{\sum_{r=1}^{p} \alpha_r - \sum_{r=1}^{q} \rho_r - \frac{1}{2}(p-q-1) - \alpha}{2} (2\pi)^{\frac{1}{2} - \frac{1}{2} m (p-q)}$$

$$\times E\left( \begin{matrix} \triangle(2m, \alpha_1), \ldots, \triangle(2m, \alpha_p) \\ \alpha, \alpha/2, \frac{\alpha+1}{2}, \triangle(2m, \rho_1), \ldots, \triangle(2m, \rho_q) \end{matrix} : \frac{4z^2}{(2m)^{2m(p-q)}} \right),$$

the path of integration being as in [4, p. 374, (36)].

**Proof**

On the left hand side of (2·1), reducing the parameters of the E-function with the help of (1·1) and (1·2), the integral can be put in the form

$$\frac{1}{2\pi i} \int \frac{\Gamma(\zeta)}{\Gamma(\alpha-\zeta)} \{z m^{-m(p-q)}\}^{\zeta} \left[ \prod_{r=1}^{q} m^{\alpha_r - \rho_r} \Gamma(\rho_r - \alpha_r) \right]^{-1}$$

$$\times \prod_{r=1}^{q} \int_0^1 \lambda_r^{\alpha_r - m\zeta - 1} (1-\lambda_r)^{\rho_r - \alpha_r - 1} d\lambda_r$$

$$\times \left[ \prod_{r=q+1}^{p} (2\pi)^{\frac{1}{2} - \frac{1}{2} m} m^{\alpha_r - m\zeta - \frac{1}{2}} \right]^{-1} \prod_{r=q+1}^{p} \int_0^{\infty} e^{-\lambda_r} \lambda_r^{\alpha_r - m\zeta - 1} d\lambda_r$$

$$\times E\left( \begin{matrix} - \\ \alpha \end{matrix} : - \frac{z}{(\lambda_1 \ldots \lambda_p)^m} \right) d\zeta$$

Here changing the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\left[ \prod_{r=1}^{p} m^{\alpha_r - \rho_r} \Gamma(\rho_r - \alpha_r) \right]^{-1} \prod_{r=1}^{q} \int_0^1 \lambda_r^{\alpha_r - 1} (1-\lambda_r)^{\rho_r - \alpha_r - 1} d\lambda_r$$

$$\times \left[ \prod_{r=q+1}^{p} (2\pi)^{\frac{1}{2} - \frac{1}{2} m} m^{\alpha_r - \frac{1}{2}} \right]^{-1} \prod_{r=q+1}^{p} \int_0^{\infty} e^{-\lambda_r} \lambda_r^{\alpha_r - 1}$$

$$\times E\left( \begin{matrix} - \\ \alpha \end{matrix} : - \frac{z}{(\lambda_1 \ldots \lambda_p)^m} \right) d\lambda_r \times \frac{1}{2\pi i} \int \frac{\Gamma(\zeta)}{\Gamma(\alpha-\zeta)} \{z m^{-m(p-q)}\}^{\zeta} \cdot \frac{m^{m(p-q)\zeta}}{(\lambda_1 \ldots \lambda_p)^{m\zeta}} d\zeta.$$

Substituting from [4, p. 374, (36)], for the last line, the expression becomes,

$$\left[ \prod_{r=1}^{q} m^{\alpha_r - \rho_r} \Gamma(\rho_r - \alpha_r) \right]^{-1} \prod_{r=1}^{q} \int_0^1 \lambda_r^{\alpha_r - 1} (1-\lambda_r)^{\rho_r - \alpha_r - 1} d\lambda_r$$

$$\times \left[ \prod_{r=q+1}^{p} (2\pi)^{\frac{1}{2} - \frac{1}{2} m} m^{\alpha_r - \frac{1}{2}} \right]^{-1} \prod_{r=q+1}^{p} \int_0^{\infty} e^{-\lambda_r} \lambda_r^{\alpha_r - 1} E\left( \begin{matrix} - \\ \alpha \end{matrix} : - \frac{z}{(\lambda_1 \ldots \lambda_p)^m} \right)$$

$$\times E\left( \begin{matrix} - \\ \alpha \end{matrix} : \frac{z}{(\lambda_1 \ldots \lambda_p)^m} \right) d\lambda_r.$$

Now replacing the product of E-functions with the help of the result

$$E\left( \begin{matrix} - \\ \alpha \end{matrix} : z \right) E\left( \begin{matrix} - \\ \alpha \end{matrix} : -z \right) = 2^{\frac{1}{2} - \alpha} (2\pi)^{\frac{1}{2}} E\left( \begin{matrix} - \\ \alpha, \alpha/2, \frac{\alpha+1}{2} \end{matrix} : 4z^2 \right),$$

which is deduced by applying [3, p. 433, (2)] to [2, p. 186, (3)].

Then on applying (1·1) and (1·2), the right hand side of (2·1) is obtained.

(*ii*) *The second integral.* If $p \geqslant q$, $|\text{amp } z| < \frac{1}{2}(mp-mq-1)\pi$, $Re\ (\rho_r-\alpha_r)>0$, $(r=1, 2, \ldots, q)$, $Re\ (\alpha_r)>0$, $(r=1, 2, \ldots, p)$, $m$ is a positive integer, $\alpha$ and $\beta$ are such that E-functions exist,

$$(2\cdot2)\quad \frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\,\Gamma(\alpha-\zeta)}{\Gamma(\beta-\zeta)}\,E\left(\begin{matrix}\alpha,\ \Delta(m,\alpha_1-m\zeta),\ldots,\Delta(m,\alpha_p-m\zeta)\\ \beta,\ \Delta(m,\rho_1-m\zeta),\ldots,\Delta(m,\rho_q-m\zeta)\end{matrix}:-z\,m^{-m(p-q)}\right)$$

$$\times\ \{z\,m^{-m(p-q)}\}^{\zeta}\,d\zeta$$

$$=\frac{\Gamma(\alpha)}{\Gamma(\beta-\alpha)}\ 2^{\sum\limits_{r=1}^{p}\alpha_r-\sum\limits_{r=1}^{q}\rho_r-\frac{1}{2}(p-q-1)-\beta}\ (2\pi)^{\frac{1}{2}-\frac{1}{2}m\,(p-q)}$$

$$\times\ E\left(\begin{matrix}\alpha,\ \beta-\alpha,\ \Delta(2m,\alpha_1),\ldots,\Delta(2m,\alpha_p)\\ (\beta,\ \beta/2,\ \frac{\beta+1}{2}\ \Delta(2m,\rho_1),\ldots,\Delta(2m,\rho_q)\end{matrix}:-\frac{4z^2}{(2m)^{2m(p-q)}}\right),$$

the path of integration being as in [4, p. 374, (36)] with loop if necessary, to ensure that $\alpha$ is to the right of the contour.

The integral can be established by applying the same procedure as in (2·1), and using the result, which is deduced by applying [3, p. 433, (2)] to (2, p. 186, (5)].

(*iii*) *The third integral.* If $p \geqslant q$, $|\text{amp } z| < \frac{1}{2}(mp-mq+1)\pi$, $Re(\rho_r-\alpha_r)>0$, $(r=1, 2, \ldots, q)$, $Re(\alpha_r)>0$, $(r=1, 2, \ldots, p)$, $m$ is a positive integer, $\alpha$ and $\beta$ are such that E-functions exists,

$$(2\cdot3)\quad \frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\,\Gamma(\alpha-\zeta)}{\Gamma(2\alpha-\zeta)}\,E\left(\begin{matrix}\beta,\ \Delta(m,\alpha_1-m\zeta),\ldots,\Delta(m,\alpha_p-m\zeta)\\ 2\beta,\ \Delta(m,\rho_1-m\zeta),\ldots,\Delta(m,\rho_q-m\zeta)\end{matrix}:-zm^{-m(p-q)}\right)$$

$$\times\ \{zm^{-m(p-q)}\}^{\zeta}\,d\zeta$$

$$=2^{\sum\limits_{r=1}^{p}\alpha_r-\sum\limits_{r=1}^{q}\rho_r-\frac{1}{2}(p-q-1)-\alpha-\beta}\ (2\pi)^{\frac{1}{2}-\frac{1}{2}m\,(p-q)}$$

$$\times\ E\left(\begin{matrix}\frac{\alpha+\beta}{2},\ \frac{\alpha+\beta+1}{2},\ \Delta(2m,\alpha_1),\ldots,\Delta(2m,\alpha_p)\\ \alpha+\frac{1}{2},\ \beta+\frac{1}{2},\ \alpha+\beta,\ \Delta(2m,\rho_1),\ldots,\Delta(2m,\rho_q)\end{matrix}:-\frac{4z^2}{(2m)^{2m(p-q)}}\right),$$

the path of integration being as in [4, p. 374, (36)] with loop if necessary to ensure that $\alpha$ is to the right of the contour.

The integral can be established by applying the same procedure as in (2·1), and using the result, which is deduced by applying [2, p. 433, (2)] to [2, p. 186,(6)].

(*iv*) *The fourth integral.* If $p \geqslant q$, $|\text{amp } z| < \frac{1}{2}(mp-mq-1)\pi$, $Re(\rho_r-\alpha_r)>0$, $(r=1, 2, \ldots, q)$, $Re(\alpha_r)>0$, $(r=1, 2, \ldots, p)$, $m$ is a positive integer, $\alpha$ and $\beta$ are such that E-functions exists,

$$(2\cdot4)\quad \frac{1}{2\pi i}\int\frac{\Gamma(\zeta)}{\Gamma(\alpha-\zeta)\,\Gamma(\beta-\zeta)}\,E\left(\begin{matrix}\Delta(m,\alpha_1-m\zeta),\ldots,\Delta(m,\alpha_p-m\zeta)\\ \alpha,\ \beta,\ \Delta(m,\rho_1-m\zeta),\ldots,(m,\rho_q-m\zeta)\end{matrix}:-zm^{-m(p-q)}\right)$$

$$\times\ \{zm^{-m(p-q)}\}^{\zeta}\,d\zeta$$

$$= \frac{\Gamma\left(\frac{\alpha+\beta-1}{2}\right)\Gamma\left(\frac{\alpha+\beta}{2}\right)}{\Gamma\left(\frac{\alpha+\beta-1}{3}\right)\Gamma\left(\frac{\alpha+\beta}{3}\right)\Gamma\left(\frac{\alpha+\beta+1}{3}\right)} 2^{\sum_{r=1}^{p}\alpha_r-\sum_{r=1}^{q}\rho_r-\frac{1}{2}(p-q-2)-\alpha-\beta} (2\pi)^{1-\frac{1}{2}m(p-q)}$$

$$\times E\left(\begin{matrix}\frac{\alpha+\beta-1}{3}, \frac{\alpha+\beta}{3}, \frac{\alpha+\beta+1}{3}, \Delta(2m, \alpha_1), \ldots, \Delta(2m, \alpha_p) : (4/3)^3 \frac{z^2}{(2m)^{2m(p-q)}} \\ \alpha, \beta, \alpha/2, \beta/2, \frac{\alpha+1}{2}, \frac{\beta+1}{2}, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta-1}{2}, \Delta(2m, \rho_1), \ldots, \Delta(2m, \rho_q)\end{matrix}\right)$$

the path of integration being as in [4, p. 374, (36)].

The integral can be established by applying the same procedure as in (2·1), and using the result, which is deduced by applying [3, p. 433, (2)] to [2, p. 186, (7)].

*Note*: In the above integrals, the restriction on $\rho$'s can be omitted, as the path of integration from 0 to 1 can be replaced by contours starting from 0 passing round 1, and returning to 0.

### 3. Summation of Series

(*i*) *The first series.* If $|\text{amp } z| < \pi/2$, $Re(\gamma)>0$,

$$(3{\cdot}1) \qquad \sum_{r=1}^{\infty} \frac{(-1)^r z^{-2r}}{(\gamma)_r\, r!} E\left(\begin{matrix}\gamma+r : z\\ \alpha+r\end{matrix}\right) E\left(\begin{matrix}\gamma+r : -z\\ \alpha+r\end{matrix}\right)$$

$$= \Gamma(\gamma)2^{\gamma-\alpha}\ E\left(\begin{matrix}\gamma/2, \frac{\gamma+1}{2} : z^2\\ \alpha, \alpha/2, \frac{\alpha+1}{2}\end{matrix}\right)$$

(*ii*) *The second series.* If $|\text{amp } z| < \pi$, $Re(\gamma)>0$, $Re(\alpha)>0$,

$$(3{\cdot}2) \qquad \sum_{r=0}^{\infty} \frac{(-1)^r z^{-2r}}{(\gamma)_r\, r!} E\left(\begin{matrix}\alpha+r, \gamma+r : z\\ \beta+r\end{matrix}\right) E\left(\begin{matrix}\alpha+r, \gamma+r : -z\\ \beta+r\end{matrix}\right)$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\gamma)}{\Gamma(\beta-\alpha)} 2^{\gamma-\beta}\ E\left(\begin{matrix}\alpha, \beta-\alpha, \gamma/2, \frac{\gamma+1}{2} : -z^2\\ \beta, \beta/2, \frac{\beta+1}{2}\end{matrix}\right).$$

(*iii*) *The third series.* If $|\text{amp } z| < \pi$, $Re(\gamma)>0$, $Re(\alpha)>0$, $Re(\beta)>0$,

$$(3{\cdot}3) \qquad \sum_{r=0}^{\infty} \frac{(-1)^r z^{-2r}}{(\gamma)_r\, r!} E\left(\begin{matrix}\alpha+r, \gamma+r : z\\ 2\alpha+r\end{matrix}\right) E\left(\begin{matrix}\beta+r, \gamma+r : -z\\ 2\beta+r\end{matrix}\right)$$

$$= 2^{\gamma-\alpha-\beta}\ E\left(\begin{matrix}\frac{\alpha+\beta}{2}, \frac{\alpha+\beta+1}{2}, \frac{\gamma}{2}, \frac{\gamma+1}{2} : -z^2\\ \alpha+\frac{1}{2}, \beta+\frac{1}{2}, \alpha+\beta\end{matrix}\right).$$

(*iv*) *The fourth series.* If $|\text{amp } z| < 0$, $Re(\gamma)>0$,

$$(3{\cdot}4) \qquad \sum_{r=0}^{\infty} \frac{(-1)^r z^{-2r}}{(\gamma)_r\, r!} E\left(\begin{matrix}\gamma+r : z\\ \alpha+r, \beta+r\end{matrix}\right) E\left(\begin{matrix}\gamma+r : -z\\ \alpha+r, \beta+r\end{matrix}\right)$$

$$= \frac{\Gamma\left(\frac{\alpha+\beta-1}{2}\right)\Gamma\left(\frac{\alpha+\beta}{2}\right)}{\Gamma\left(\frac{\alpha+\beta-1}{3}\right)\Gamma\left(\frac{\alpha+\beta}{3}\right)\Gamma\left(\frac{\alpha+\beta+1}{3}\right)} \pi^{\frac{1}{2}}\, 2^{1+\gamma-\alpha-\beta}$$

$$\times E\left(\begin{array}{l} \frac{\alpha+\beta+1}{3}, \frac{\alpha+\beta}{3}, \frac{\alpha+\beta-1}{3}, \frac{\gamma}{2}, \frac{\gamma+1}{2} \quad : 16/27\, z^2 \\ \alpha, \beta, \alpha/2, \beta/2, \frac{\alpha+1}{2}, \frac{\beta+1}{2}, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta-1}{2} \end{array}\right)$$

**Proof**

To proof (3·1), substituting on the left from [4, p. 374, (36)], we get

$$\sum_{r=0}^{\infty} \frac{(-1)^r z^{-2r}}{(\gamma)_r\, r\,!} \times \frac{1}{2\pi i} \int \frac{\Gamma(\zeta)\, \Gamma(\gamma+r-\zeta)}{\Gamma(\alpha+r-\zeta)} z^{\zeta}\, d\zeta \times \frac{1}{2\pi i} \int \frac{\Gamma(\eta)\, \Gamma(\gamma+r-\eta)}{\Gamma(\alpha+r-\eta)} (-z)^{\eta} d\eta$$

Here replacing $\zeta$ and $\eta$ by $\zeta+r$ and $\eta+r$ respectively, and changing the order of integration and sumation, which is justified by the methods given in [1, p. 500], then the expression becomes

$$\frac{1}{2\pi i} \int \frac{\Gamma(\zeta)\, \Gamma(\gamma-\zeta)}{\Gamma(\alpha-\zeta)} z^{\zeta}\, d\zeta \times \frac{1}{2\pi i} \int \frac{\Gamma(\eta)\, \Gamma(\gamma-\eta)}{\Gamma(\alpha-\eta)} (-z)^{\eta}\ {}_2F_1\left(\begin{array}{l} \zeta, \eta\, ;\, 1 \\ \ :\ \gamma \end{array}\right) d\eta$$

Now applying Gauss's theorem, we have

$$\frac{\Gamma(\gamma)}{2\pi i} \int \frac{\Gamma(\zeta)}{\Gamma(\alpha-\zeta)} z^{\zeta}\, E\left(\begin{array}{l} \gamma-\zeta \ :-z \\ \alpha \end{array}\right) d\zeta.$$

Us ng (2·1), with $m=1$, $p=1$, $q=0$, the result (3·1) follows.

The series (3·2), (3·3) and (3·4) can be established by applying the same procedure as above and using (2·2), (2·3) and (2·4) respectively.

The validity of the expansions (3·1), (3·2), (3·3) and (3·4) is justified under the stated conditions by the procedure followed to establish them.

**Acknowledgements**

I am thankful to Dr. V. M. Bhise fcr his guidance in the preparation of this paper. I am also thankful to the referee for some useful suggessions for revising the paper. My thanks are also due to Principal Dr. S. M. Das Gupta for the facilities he gave to me.

**References**

1. Bromwich, T. J. I. Theory of infinite series. *New York*, (1959).
2. Erdelyi, A. Higher transcendental functions. *New York*, **Vol. 1** : (1953).
3. Erdelyi A. Tables of integral transforms. *New York*, **Vol. 2** : (1954).
4. MacRobert, T. M. Functions of a Complex variable. *London*, (1962).
5. MacRobert, T. M. Integration of E-functions with respect to their parameters. *Proc. Glasgow Math. Assoc.*, **5** : 118–120, (1962).
6. Ragab, F. M. Integration of E-functions with respect to their parameters. *Proc. Glasgow Math. Assoc*, **3** : 94–98, (1956).
7. Ragab, F. M. Integration of E-functions and related functions with respect to their parameters. *Koninkl. Nederl. Akademic Van Wetenschappen–Amsterdam, Proc. A.* **61** : 335–340. (1958).
8. Ragab, F. M. Summation of a series of products of E-function. *Proc. Glasgow Math. Assoc.* **5** : 118–120, (1962).

# A note on the existence of repeated and non-repeated zeros of a transcendental function

*By*

S. R. MUKHERJEE *and* K. N. BHOWMICK

*Engineering College, Banaras Hindu University*

[Received on 24th August, 1966]

**Abstract**

Recently[1] it has been shwon that the zeros of the transcendental function $G(x) \equiv x J_\nu'(x) + f(x) J_\nu(x)$ associated with the circular functions $f(x) \equiv \cos x, \sin x$ are real, as a direct consequence of the Lommel's integral result which has been applied previously by Watson[2] in testifying the existence of real zeros of the Bessel function $J_\nu(x)$, the transcendental function $x J_\nu'(x) + h J_\nu(x)$ ($h \equiv$ a constant) and the Bessel function of the second kind $Y_0(x)$.

The object of the present work is to examine the existence of repeated and imaginary zeros of the function $G(x)$ associated with the circular functions $\sin x$, $\cos x$, the Struve functions $x^\mu H_\mu(x)$, $x^{-\mu} H_\mu(x)$ and the expression $(Ax^2 + B)$ by the aid of Lommel's integral result.

## 1. Existence of repeated and non-repeated roots of the equation $G(x) = 0$

By virtue of the result

$$\int_0^1 t\, J_\nu^2(xt)\, dt = -\frac{1}{2x} \begin{vmatrix} J_\nu(x) & x\, J_\nu'(x) \\ J_\nu'(x) & \frac{d}{dx}[x J_\nu'(x)] \end{vmatrix}, \tag{1·1}$$

given by Watson[3], we find that

$$\int_0^1 t\, J_\nu^2(xt)\, dt = -\frac{1}{2x} \begin{vmatrix} J_\nu(x) & -f(x)\, J_\nu(x) \\ \frac{1}{x} J_\nu(x) f(x) & -\frac{d}{dx} f(x)\, J_\nu(x) \end{vmatrix} = \frac{1}{2x} J_\nu^2(x) f'(x) \tag{1·2}$$

subject to the assumption that $G(x) = G'(x) = 0$.

On the other hand if $G(x) \neq G'(x)$ and $G(x) = 0$, then (1·1) assumes the form

$$\int_0^1 t\, J_\nu^2(xt)\, dt = \frac{1}{2x^2}\left[ (x^2 - \nu^2) J_\nu^2(x) + x^2 J_\nu'^2(x) \right]$$

$$= \frac{1}{2x^2} J_\nu^2(x) \left[ f^2(x) + x^2 - \nu^2 \right] \tag{1·3}$$

Again, on putting $x = i\beta$, (1·2) and (1·3) may be expressed in the forms

$$\int_0^1 t J_\nu^2(i\beta t)\, dt = \int_0^1 t \sum_{m=0}^\infty \left\{ \frac{(\beta t/2)^{2\nu+2m}\, \Gamma(2\nu + 2m + 1)}{\underline{|m}\, \Gamma(2\nu + m + 1)\, \{\Gamma(\nu + m + 1)\}^2} \right\} dt$$

$$= \frac{f'(i\beta)}{2i\beta} \sum_{m=0}^\infty \frac{(\beta/2)^{2\nu+2m}\, \Gamma(2\nu + 2m + 1)}{\underline{|m}\, \Gamma(2\nu + m + 1)\, \{\Gamma(\nu + m + 1)\}^2} \tag{1·4}$$

$$\int_0^1 t \sum_{m=0}^{\infty} \left\{ \frac{(\beta t/2)^{2\nu+2m}\,\Gamma(2\nu+2m+1)}{\lfloor m\ \Gamma(2\nu+m+1)\ \{\Gamma(\nu+m+1)\}^2} \right\} dt$$

$$= -\frac{[f^2(i\beta)-\beta^2-\nu^2]}{2\beta^2} \sum_{m=0}^{\infty} \frac{(\beta/2)^{2\nu+2m}\,\Gamma(2\nu+2m+1)}{\lfloor m\,\Gamma(2\nu+m+1)\ \{\Gamma(\nu+m+1)\}^2} \qquad (1\cdot5)$$

respectively by virtue of the result

$$J_\nu^2(x) = \sum_{m=0}^{\infty} \frac{(-)^m\,(x/2)^{2\nu+2m}\cdot\Gamma(2\nu+2m+1)}{\lfloor m\ \Gamma(2\nu+m+1)\ \{\Gamma(\nu+m+1)\}^2}$$

given by Watson[4].

Obviously for the consistency of the relations, the right hand sides (1·2), (1·3), (1·4) and (1·5) should always be positive, and hence it may be concluded that

(1-*a*) The real and imaginary roots of the equation $G(x) = 0$ are repeated or non-repeated according as $\frac{1}{x} f'(x) \gtrless 0$; with the addition that $f'(x) = 0$ for the existence of non-repeated roots only.

(1-*b*) The real and imaginary roots of $G(x) = 0$ are repeated or non-repeated according as $(1/x^2)\,[f^2(x) - x^2 - \nu^2] \lesseqgtr 0$, with the addition that $f^2(x) = \nu^2 - x^2$ for the existence of repeated roots only.

(1-*c*) It is worth mentioning that the roots of the equation $G(x) = 0$, associated with a constant $f(x) \equiv h$ are non-repeated, as observed by Watson[3], forms a special case of (1-*a*).

We shall now investigate the existence of repeated and non-repeated roots of $G(x) = 0$, associated with certain special functions, in the light of above observations.

(1-*d*) **Applications.**

**Example 1**

Positive roots of the equation $G(x) = 0$, associated with odd function $f(x) \equiv \sin x$, are repeated or non-repeated according as

$$0 < x < \frac{\pi}{2} \quad \text{or} \quad \frac{\pi}{2} < x < \frac{3\pi}{2};$$

whereas negative roots behave in the opposite manner.

For,

$$\left.\begin{aligned} \frac{f'(x)}{x} &= \frac{\cos x}{x} > 0 \\ \frac{f'(x)}{x} &= \frac{\cos x}{x} < 0 \end{aligned}\right\rangle \left\{\begin{aligned} 0 < x < \frac{\pi}{2} \\ \frac{\pi}{2} < x < \frac{3\pi}{2} \end{aligned}\right\}$$

**Example 2**

Positive and negative roots of $G(x) = 0$ associated with the even function $f(x) \equiv \cos x$ remain non-repeated for all '$x$' satisfying $0 < x < \pi$.

The result is obviously true by virtue of (1-*a*).

**Example 3**

(*i*) Positive roots of $G(x) = 0$ associated with Struve function $x^{\mu}$ $H_{\mu}(x)$ of order $\mu$ are repeated or non-repeated according as $\mu \gtrless \frac{3}{2}$.

For

$$f'(x) = \frac{d}{dx}\,[x^{\mu} H_{\mu}(x)\,] = x^{\mu} H_{\mu-1}(x) \gtrless^{(6)} 0 \text{ according as } \mu \gtrless \tfrac{3}{2} \text{ as given}$$

by Watson[6].

Positive roots of $G(x) = 0$, associated with the Struve function $f(x) \equiv x^{-\mu}$ $H_{\mu}(x)$, are repeated when $\mu < -\frac{1}{2}$ and the larger roots remain non-repeated for $\mu > -\frac{1}{2}$

Obviously, when $x$ is not large,

$$f'(x) = \frac{d}{dx}\,[x^{-\mu} H_{\mu}(x)] = \left[\frac{2^{-\mu}\,\pi^{-\frac{1}{2}}}{\Gamma(\mu+\frac{3}{2})} - x^{-\mu}\,H_{\mu+1}\,(x)\;\right]^{(5)} > 0,$$

provided that $\mu < -\frac{1}{2}$, which establishes the existence of repeated roots of $G(x) = 0$.

On the other hand, when $x$ is large, we find

$$f'(x) = \frac{d}{dx}\,[x^{-\mu} H_{\mu}(x)] \sim 0,$$

by virtue of the result $H_{\mu+1}(x) \sim (x/2)^{\mu}/\sqrt{\pi}\,\Gamma(\mu+\frac{3}{2})$ for large values of $x$ and which proves the existence of non-repeated roots of $G(x) = 0$.

**Example 4**

Roots of $G(x) = 0$, associated with the function $f(x) = \sqrt{\nu^2 - x^2}$ are repeated.

For, by virtue of (1·3) and (1·5), we find that

$$\int_0^1 t\,J_{\nu}^{2}\,(xt)\;dt = 0,$$

for real or imaginary values of $x$; which leads to a contradiction.

**Example 5**

Roots of the equation $G(x) = 0$, associated with the even function $f(x) = Ax^2 + B$ are repeated or non-repeated according as $A > 0$ or $A \leqslant 0$.

The result is obviously true by virtue of (1·2).

**Acknowledgment**

The authors are grateful to Principal M. Sen Gupta for his encouragement in the preparation of this note.

**References**

1. Mukherjee, S. R. and Bhowmick, K. N. On zeros of a transcendental function associated with Bessel functions of the first kind of orders $\nu$ and $\nu + 1$. *Accepted for publication in Proc. Nat. Acad. Sci. (India)*, 36(A) : (1966).
2. Watson, G. N. Theory of Bessel functions, *Cambridge*, 2nd Ed., 482–483, (1958).
3. Watson, G. N. *Ibid*, 480, (1958).
4. Watson, G. N. *Ibid.*, 147, (1958).
5. Watson, G. N. *Ibid.*, 329, (1958).
6. Watson, G. N. *Ibid.*, 337, (1958).

# Certain Integrals involving the Product of Two Generalised Hypergeometric Polynomials

*By*

MANILAL SHAH

*Department of Mathematics, P. M. B. G. College, Indore*

[Received on 24th August, 1966]

**Abstract**

In this paper, using a generalised hypergeometric polynomial defined by

$$F_n(x) = x^{(\delta-1)n}\,{}_{p+\delta}F_q\left[\begin{matrix}\Delta(\delta,-n),\, a_1,\, a_2,\, \ldots\ldots,\, a_p\,; \\ b_1,\, b_2,\, \ldots\ldots,\, b_q\,;\end{matrix}\ \mu x^k\right]$$

where $\triangle\ (\delta, -n)$ represents a set of parameters :

$$\frac{-n}{\delta},\ \frac{-n+1}{\delta},\ \ldots\ldots,\ \frac{-n+\delta-1}{\delta}$$

and $\delta$, $n$ are positive integers, we have derived two integrals involving the product of two hypergeometric polynomials. A number of known and unknown cases of the integrals have been obtained.

## 1. Introduction

The object of this paper is to evaluate the product of two generalised hypergeometric polynomials in series and evaluate two integrals involving the product of hypergeometric polynomials. A finite integral is given in section 3 and the infinite integral in section 4. The integrals are in a generalised form and on specialising the parameters yield many known as well as unknown results.

For ease in writing, we employ the contracted notation,

$${}_pF_q(x) = {}_pF_q\left(\begin{matrix}a_p \\ b_q\end{matrix}\,\middle|\, x\right) = \sum_{k=0}^{\infty} \frac{(a_p)_k\, x^k}{(b_q)_k\, \lfloor k}$$

Thus $(a_p)_k$ is to be interpreted as $\prod_{j=1}^{p} (a_j)_k$ and similarly $(b_q)_k$.

2. We define the generalised hypergeometric polynomial in the form

$$F_n(x) = x^{(\delta-1)n}\,{}_{p+\delta}F_q\left[\begin{matrix}\Delta(\delta,-n),\, a_p\,; \\ b_q\,;\end{matrix}\ \mu x^k\right] \tag{2·1}$$

where $\triangle(\delta, -n)$ represents a set of $\delta$ parameters,

$$\frac{-n}{\delta},\ \frac{-n+1}{\delta},\ \ldots\ldots,\ \frac{-n+\delta-1}{\delta}$$

and $\delta$, $n$ are positive integers.

The polynomial is in a generalised form and yields many known polynomials by particular choice of parameters.

Particular cases of the generalised hypergeometric polynomial :—

(*a*) Consider the polynomial with $\delta = \mu = k = 1$

(*i*) Setting $a_1 = n + \alpha + \beta + 1$, $b_1 = 1 + \alpha$, $b_2 = \frac{1}{2}$, we obtain

$$F_n(x) = {}_{p+1}F_q \left[ \begin{matrix} -n, n + \alpha + \beta + 1, a_2, \ldots, a_p ; \\ 1 + \alpha, \frac{1}{2}, b_3, \ldots, b_q ; \end{matrix} x \right] \tag{2·2}$$

$$= \frac{\underline{|n}}{(1 + \alpha)_n} f_n^{(\alpha,\beta)} \left( \begin{matrix} a_2, \ldots, a_p ; \\ b_3, \ldots, b_q ; \end{matrix} x \right)$$

a generalised Sister Celine's polynomial which reduces to a Sister Celine's polynomial [1, p. 806] when $\alpha = \beta = 0$.

(*ii*) With $p = q = 2$, $a_1 = n + \alpha + \beta + 1$, $a_2 = \rho$,
$b_1 = 1 + \alpha$, $b_2 = \sigma$,

we get

$$F_n(x) = {}_3F_2 \left[ \begin{matrix} -n, n + \alpha + \beta + 1, \rho ; \\ 1 + \alpha, \sigma ; \end{matrix} x \right] \tag{2·3}$$

$$= \frac{\underline{|n}}{(1 + \alpha)_n} H_n^{(\alpha, \beta)} (\rho, \sigma, x)$$

a generalised Rice's polynomial ( [2], p. 158, equation (2·3) ). It reduces to a Rice's polynomial when $\alpha = \beta = 0$.

(*iii*) Substituting $p = q = 1$, $a_1 = n + \alpha + \beta + 1$, $b_1 = 1 + \alpha$,
we have

$$F_n(x) = {}_2F_1 \left[ \begin{matrix} -n, n + \alpha + \beta + 1 ; \\ 1 + \alpha \quad ; \end{matrix} x \right]$$

$$= \frac{\underline{|n}}{(1 + \alpha)_n} P_n^{(\alpha, \beta)} (1 - 2x)$$

a Jacobi polynomial which can be reduced to either Gegenbauer, Legendre or Tchebicheff polynomials by specialising the parameters.

(*iv*) With $p = 0$, $q = 1$, $b_1 = 1 + \alpha$, we obtain

$$F_n(x) = {}_1F_1 \left[ \begin{matrix} -n ; \\ 1 + \alpha ; \end{matrix} x \right] = \frac{\underline{|n}}{(1 + \alpha)_n} L_n^{(\alpha)} (x)$$

a generalised Laguerre polynomial.

(*v*) Taking $p = 1$, $q = 2$, $a_1 = 2\gamma + n$, $b_1 = \gamma + \frac{1}{2}$, $b_2 = 1 + b$, we get

$$F_n(x) = {}_2F_2 \left[ \begin{matrix} -n, 2\gamma + n ; \\ \gamma + \frac{1}{2}, 1 + b ; \end{matrix} x \right]$$

one of the generalizations [3] of the Bessel polynomials which with $\gamma = \frac{1}{2}$, $b = 0$, reduces to a Bateman's polynomial $Z_n(x)$.

(*b*) Consider the polynomial with $\delta = 2$, $k = -2$.

($i$) Setting $p = q = 0$, $\mu = -1$, we obtain

$$F_n(x) = x^n \,{}_2F_0 \left[ \begin{matrix} \frac{-n}{2}, \frac{-n+1}{2} ; & -x^{-2} \\ - ; & \end{matrix} \right] = \frac{1}{2^n} H_n(x)$$

a Hermite polynomial.

($ii$) Substituting $p = 1$, $q = 2$, $a_1 = \gamma - \beta$, $b_1 = \gamma$, $b_2 = 1 - \beta - n$, $\mu = 1$, we get

$$F_n(x) = x^n \,{}_3F_2 \left[ \begin{matrix} \frac{-n}{2}, \frac{-n+1}{2}, \gamma - \beta ; & x^{-2} \\ \gamma, 1 - \beta - n ; & \end{matrix} \right] = \frac{\lfloor n}{(\beta)_n 2^n} R_n(\beta, \gamma ; x)$$

a Bedient's polynomial ([4]).

**3.** Let us consider the product of two generalised hypergeometric polynomials (2·1).

$$F_n(x) F_m(x) = x^{(\delta-1)n} \,{}_{p+\delta}F_q \left[ \begin{matrix} \Delta(\delta, -n), a_p ; & \mu x^c \\ b_q ; & \end{matrix} \right] x^{(\gamma-1)m} \,{}_{l+\gamma}F_k \left[ \begin{matrix} \Delta(\gamma, -m), \rho_l ; & \lambda x^d \\ \sigma_k ; & \end{matrix} \right]$$

$$= x^{(\delta-1)n+(\gamma-1)m} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left( \frac{-n+i}{\delta} \right)_r (a_p)_r \mu^r x^{cr}}{\lfloor r \, (b_q)_r}$$

$$\times \frac{\prod_{i=0}^{\gamma-1} \left( \frac{-m+i}{\gamma} \right)_s (\rho_l)_s \lambda^s x^{ds}}{\lfloor s \, (\sigma_k)_s}$$

$$= x^{(\delta-1)n+(\gamma-1)m} \sum_{r=0}^{\infty} \sum_{s=0}^{r} \frac{\prod_{i=0}^{\delta-1} \left( \frac{-n+i}{\delta} \right)_{r-s} \prod_{i=0}^{\gamma-1} \left( \frac{-m+i}{\gamma} \right)_s (a_p)_{r-s} (\rho_l)_s}{\lfloor s \, \lfloor r-s \, (b_q)_{r-s} (\sigma_k)_s}$$

$$\times \mu^{r-s} \lambda^s x^{c(r-s)+ds}.$$

Now using the formula

$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k} ; \; 0 \leqslant k \leqslant n,$$

we obtain

$$(3·1) \qquad F_n(x) F_m(x) = x^{(\delta-1)n+(\gamma-1)m} \sum_{r=0}^{\infty} \sum_{s=0}^{r} \frac{\prod_{i=0}^{\delta-1} \left( \frac{-n+i}{\delta} \right)_r \prod_{i=0}^{\gamma-1} \left( \frac{-m+i}{\gamma} \right)_s}{\lfloor r \, \lfloor s \, \prod_{i=0}^{\delta-1} \left( 1 + \frac{n-i}{\delta} - r \right)_s}$$

$$\times \frac{(a_p)_r (1-b_q-r)_s (\rho_l)_s (-r)_s}{(b_q)_r (1-a_p-r)_s (\sigma_k)_s} \left( \frac{\lambda}{\mu} \right)^s \mu^r (-1)^{(p+q+\delta+1)s} x^{c(r-s)+ds}$$

$$(3\cdot 2)\qquad F_n(x)\,F_m(x) = x^{(\delta-1)n+(\gamma-1)m}\sum_{r=0}^{\infty}\frac{\prod_{i=0}^{\delta-1}\left(\frac{-n+i}{\delta}\right)_r (a_p)_r\,\mu^r\,x^{cr}}{\lfloor r\,(b_q)_r}$$

$$\times\ {}_{\gamma+l+q+1}F_{p+k+\delta}\left(\begin{matrix}\frac{-m}{\gamma},\ldots\ldots,\frac{-m+\gamma-1}{\gamma},\ \rho_l,\ 1-b_q-r,\ -r\\ 1+\frac{n}{\delta}-r\ldots\ldots,1+\frac{n-\delta+1}{\delta}-r,\ \sigma_k,\ 1-a_p-r\end{matrix}\ \middle|\ \frac{(-1)^{p+q+\delta+1}\,\lambda\,x^{d-c}}{\mu}\right)$$

with the help of (3·1), we have an integral involving the product of two generalised hypergeometric polynomials in series.

$$(3\cdot 3)\qquad \int_0^1 x^{L+(\delta-1)n+(\gamma-1)m-1}\,(1-x)^{M-1}\ {}_{p+\delta}F_q\left[\begin{matrix}\triangle(\delta,-n),\ a_p\,;\\ b_q\,;\end{matrix}\ \mu x^c\right]$$

$$\times\ {}_{l+\gamma}F_k\left[\begin{matrix}\triangle(\gamma,-m),\ \rho_l\,;\\ \sigma_k\,;\end{matrix}\ \lambda x^d\right]dx$$

$$=\Gamma(M)\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{\prod_{i=0}^{\delta-1}\left(\frac{-n+i}{\delta}\right)_r\prod_{i=0}^{\gamma-1}\left(\frac{-m+i}{\gamma}\right)_s (a_p)_r\,(\rho_l)_s\,(1-b_q-r)_s\,(-r)_s\,\mu^r}{\lfloor r\,\lfloor s\,\prod_{i=0}^{\delta-1}\left(1+\frac{n-i}{\delta}-r\right)_s (1-a_p-r)_s\,(\sigma_k)_s\,(b_q)_r}$$

$$\times\left(\frac{\lambda}{\mu}\right)^s(-1)^{(p+q+\delta+1)s}\ \frac{\Gamma\{(\delta-1)n+(\gamma-1)m+(d-c)s+cr+L\}}{\Gamma\{(\delta-1)n+(\gamma-1)m+(d-c)s+cr+L+M\}}$$

$$Re(L)>(1-\delta)n+(1-\gamma)m,\qquad Re(M)>0.$$

### 3·1. Particular cases :

Substituting in (3·3), $p=q=1$, $\delta=\gamma=1$, $d=m_1$, $c=1$, $a_1=n+\alpha+\beta+1$, $b_1=1+\alpha$, $\mu=1$, $\lambda=-z$, $\sigma_1=-m$, $L=\sigma+1$, $M=\beta+1$, and multiplying both sides by

$\frac{(1+\alpha)_n}{\lfloor n}$, we obtain

$$(3\cdot 4)\qquad \int_0^1 x^{\sigma}(1-x)^{\beta}\,P_n^{(\alpha,\beta)}(1-2x)\ {}_lF_{k-1}\left[\begin{matrix}\rho_1,\ldots\ldots,\rho_l\,;\\ \sigma_2,\ldots\ldots,\sigma_k\,;\end{matrix}\ -zx^{m_1}\right]dx$$

$$=\frac{(1+\alpha)_n\,\Gamma(1+\beta)}{\lfloor n}\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{(-n)_r\,(n+\alpha+\beta+1)_r\,(\rho_l)_s\,(-\alpha-r)_s\,(-r)_s\,(-z)^s}{\lfloor r\,\lfloor s\,(1+\alpha)_r\,(1+n-r)_s\,(-n-\alpha-\beta-r)_s\prod_{j=2}^{k}(\sigma_j)_s}$$

$$\times\frac{\Gamma\{(m_1-1)s+r+\sigma+1\}}{\Gamma\{(m_1-1)s+r+\sigma+\beta+2\}}$$

$$Re(\sigma)>-1,\ Re(\beta)>-1.$$

Replacing $r$ by $r+s$, and with the help of the relations $(\alpha)_{n+k}=(\alpha)_k\,(\alpha+k)_n$ and $\frac{(A+r)_s}{(1-A-r-s)_s}=(-1)^s$, the right hand side of (3·4) reduces to

$$(3{\cdot}5)\qquad \frac{(1+\alpha)_n\,\Gamma(1+\beta)}{|\underline{n}}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-n)_r\,(n+\alpha+\beta+1)_r\,(\rho_l)_s\,(-z)^s\,\Gamma\{m_1 s+r+\sigma+1\}}{|\underline{r}\;|\underline{s}\;(1+\alpha)_r\prod_{j=2}^{k}(\sigma_j)_s\,\Gamma\{m_1 s+r+\sigma+\beta+2\}}$$

Using multiplication formula for Gamma function in (3·5), we obtain

$$(3{\cdot}6)\qquad \frac{(1+\alpha)_n\,\Gamma(1+\beta)}{|\underline{n}\;m_1^{\beta+1}}\sum_{r=0}^{\infty}\frac{(-n)_r\,(n+\alpha+\beta+1)_r\prod_{i=1}^{m_1}\Gamma\left(\frac{\sigma+r+i}{m_1}\right)}{|\underline{r}\;(1+\alpha)_r\prod_{i=1}^{m_1}\Gamma\left(\frac{\sigma+\beta+r+1+i}{m_1}\right)}\,.$$

$$\times\;{}_{l+m_1}F_{m_1+k+1}\left(\begin{matrix}\rho_l,\ \triangle(m_1,\ \sigma+r+1)\ ;\\ \sigma_2,\ \ldots,\ \sigma_k,\ \triangle(m_1,\ \sigma+r+\beta+2)\ ;\end{matrix}\ -z\right)$$

Expressing the hypergeometric function into contour integral of Barnes' type, interchanging the order of summation and integration, and using multiplication formula, (3·6) reduces to

$$\frac{(1+\alpha)_n\,\Gamma(1+\beta)\prod_{j=2}^{k}\Gamma(\sigma_j)}{|\underline{n}\;\Gamma(\rho_l)}\;\frac{1}{2\pi i}\int_L\frac{(z)^s\,\Gamma(-s)\,\Gamma(\rho_l+s)\,\Gamma(m_1 s+\sigma+1)}{\prod_{j=2}^{k}\Gamma(\sigma_j+s)\,\Gamma(m_1 s+\sigma+\beta+2)}$$

$$\times\;{}_3F_2\left(\begin{matrix}-n,\ n+\alpha+\beta+1,\ m_1 s+\sigma+1\ ;\\ 1+\alpha,\ m_1 s+\sigma+\beta+2\ ;\end{matrix}\ 1\right)ds.$$

With the help of Saalschutz's theorem [5, p. 188] and multiplication formula, we obtain

$$\frac{\prod_{j=2}^{k}\Gamma(\sigma_j)\,(-1)^n\,\Gamma(1+\beta+n)}{|\underline{n}\;\Gamma(\rho_l)\,m_1^{\beta+1}}$$

$$\times\;\frac{1}{2\pi i}\int_L\frac{\Gamma(-s)\,\Gamma(\rho_l+s)\prod_{i=1}^{m_1}\Gamma\left(\frac{\sigma+i}{m_1}+s\right)\prod_{i=1}^{m_1}\Gamma\left(\frac{\sigma-\alpha+i}{m_1}+s\right)z^s}{\prod_{j=2}^{k}\Gamma(\sigma_j+s)\prod_{i=1}^{m_1}\Gamma\left(\frac{\sigma-n-\alpha+i}{m_1}+s\right)\prod_{i=1}^{m_1}\Gamma\left(\frac{\sigma+\beta+n+1+i}{m_1}+s\right)}ds$$

$$=\frac{\prod_{j=2}^{k}\Gamma(\sigma_j)\,\Gamma(1+\beta+n)}{\Gamma(\rho_l)\;|\underline{n}\;m_1^{\beta+1}}$$

$$\times\;\frac{1}{2\pi i}\int_L\frac{\Gamma(-s)\,\Gamma(\rho_l+s)\prod_{i=1}^{m_1}\Gamma\left(\frac{\sigma+i}{m_1}+s\right)\prod_{i=0}^{m_1-1}\Gamma\left(\frac{-\sigma+n+\alpha+i}{m_1}-s\right)z^s}{\prod_{j=2}^{k}\Gamma(\sigma_j+s)\prod_{i=1}^{m_1}\Gamma\left(\frac{\sigma+\beta+1+n+i}{m_1}+s\right)\prod_{i=0}^{m_1-1}\Gamma\left(\frac{-\sigma+\alpha+i}{m_1}+s\right)}ds$$

$$=\frac{\prod_{j=2}^{k}\Gamma(\sigma_j)\,\Gamma(1+\beta+n)}{\Gamma(\rho_l)\;|\underline{n}\;m_1^{\beta+1}}$$

$$\times\;G^{1+m_1,\ l+m_1}_{l+2m_1,\ k+2m_1}\left(z\ \middle|\ \begin{matrix}1-\rho_1,\ \ldots,\ 1-\rho_l,\ \triangle(m_1,-\sigma)\ ;\ \triangle(m_1,\alpha-\sigma)\\ 0,\triangle(m_1,\alpha-\sigma+n)\ ;\ \triangle(m_1,-\beta-\sigma-n-1),\ 1-\sigma_2,\ \ldots,\ 1-\sigma_k\end{matrix}\right)$$

on using $\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$ and the definition of $G$-function. [5, p, 206, 5·3].

This is a known result ([6], p. 200, equation (5·2)).

**3.2.** In (3·3), substituting $\delta = \gamma = 1$, $c = d = 1$, replacing $r$ by $r+s$, and using the relations $(\alpha)_{n+k} = (\alpha)_k\,(\alpha+k)_n$,

$\dfrac{(A+r)_s}{(1-A-r-s)_s} = (-1)^s$, we have

$$(3\cdot 7) \qquad \int_0^1 x^{L-1}\,(1-x)^{M-1}\; {}_{p+1}F_q\!\left[\begin{matrix} -n,\ a_p\ ; \\ b_q\ ; \end{matrix}\ \mu x\right]\, {}_{l+1}F_k\!\left[\begin{matrix} -m,\ \rho_l\ ; \\ \sigma_k\ ; \end{matrix}\ \lambda x\right] dx$$

$$= \frac{\Gamma(L)\,\Gamma(M)}{\Gamma(L+M)} \sum_{r=0}^{\infty} \frac{(-n)_r\,(a_p)_r\,(L)_r\,\mu^r}{\lfloor r\;(b_q)_r\,(L+M)_r}$$

$$\times\; {}_{l+2}F_{k+1}\left(\begin{matrix} -m,\ \rho_l,\ L+r\ ; \\ \sigma_k,\ L+M+r\ ; \end{matrix}\ \lambda\right)$$

$Re(L) > 0$, $Re(M) > 0$.

Putting $x = \dfrac{\mathrm{x}}{1+\mathrm{x}}$, we obtain

$$(3\cdot 8) \qquad \int_0^\infty \mathrm{x}^{L-1}\,(1+\mathrm{x})^{-(L+M)}\; {}_{p+1}F_q\!\left[\begin{matrix} -n,\ a_p\ ; \\ b_q\ ; \end{matrix}\ \frac{\mu\mathrm{x}}{1+\mathrm{x}}\right]\, {}_{l+1}F_k\!\left[\begin{matrix} -m,\ \rho_l\ ; \\ \sigma_k ; \end{matrix}\ \frac{\lambda\mathrm{x}}{1+\mathrm{x}}\right] d\mathrm{x}$$

$$= \frac{\Gamma(L)\,\Gamma(M)}{\Gamma(L+M)} \sum_{r=0}^{\infty} \frac{(-n)_r\,(a_p)_r\,(L)_r\,\mu^r}{\lfloor r\;(b_q)_r\,(L+M)_r}\; {}_{l+2}F_{k+1}\left(\begin{matrix} -m,\ \rho_l,\ L+r\ ; \\ \sigma_k,\ L+M+r\ ; \end{matrix}\ \lambda\right)$$

Setting $p = q = 1$, $a_1 = n+\alpha+\beta+1$, $b_1 = 1+\alpha$, $L = \xi$, $M = p - \xi$ and $m = 0$ in (3·8) and multiplying both sides by $\dfrac{(1+\alpha)_n}{\lfloor n}$, we get a known result ([2], p. 158, equation (3·1)).

**Particular cases :**

In (3·7), with $a_1 = n+\alpha+\beta+1$, $b_1 = 1+\alpha$, $b_2 = \frac{1}{2}$,
$\rho_1 = m+\gamma+\delta+1$, $\sigma_1 = 1+\gamma$, $\sigma_2 = \frac{1}{2}$,

and multiplying both sides by $\dfrac{(1+\alpha)_n\,(1+\gamma)_m}{\lfloor n\;\lfloor m}$, we obtain an integral involving the product of two generalised Sister Celine's polynomials (2·2),

$$(3\cdot 9) \quad \int_0^1 x^{L-1}\,(1-x)^{M-1}\; f_n^{(\alpha,\beta)}\left(\begin{matrix} a_2, \ldots, a_p\ ; \\ b_3, \ldots, b_q\ ; \end{matrix}\ \mu x\right) f_m^{(\gamma,\delta)}\left(\begin{matrix} \rho_2, \ldots, \rho_l\ ; \\ \sigma_3, \ldots, \sigma_k\ ; \end{matrix}\ \lambda x\right) dx$$

$$= \frac{\Gamma(L)\,\Gamma(M)\,(1+\alpha)_n\,(1+\gamma)_m}{\Gamma(L+M)\,\underline{|n}\;\underline{|m}} \sum_{r=0}^{\infty} \frac{(-n)_r\,(n+\alpha+\beta+1)_r \prod_{j=2}^{p} (a_j)_r\,(L)_r\,\mu^r}{\underline{|r}\,(1+\alpha)_r\,(\frac{1}{2})_r \prod_{j=3}^{q} (b_j)_r\,(L+M)_r}$$

$$\times\ {}_{l+2}F_{k+1}\left(\begin{matrix} -m,\ m+\gamma+\delta+1,\ \rho_2,\ \ldots,\ \rho_l,\ L+r\,; \\ 1+\gamma,\ \frac{1}{2},\ \qquad \sigma_3,\ \ldots,\ \sigma_k,\ L+M+r\,; \end{matrix}\ \lambda\right)$$

$Re(L) > 0,\ Re(M) > 0.$

(*a*) When $m = 0$, we have an integral involving a generalised Sister Celine's polynomial,

$$(3{\cdot}10) \qquad \int_0^1 x^{L-1}\,(1-x)^{M-1}\, f_n^{(\alpha,\beta)}\left(\begin{matrix} a_2,\ \ldots,\ a_p\,; \\ b_3,\ \ldots,\ b_q\,; \end{matrix}\ \mu x\right) dx$$

$$= \frac{(1+\alpha)_n\,\Gamma(L)\,\Gamma(M)}{\underline{|n}\;\Gamma(L+M)}\ {}_{p+2}F_{q+1}\left(\begin{matrix} -n,\ n+\alpha+\beta+1,\ a_2,\ \ldots,\ a_p,\ L\,; \\ 1+\alpha,\ \frac{1}{2},\ \qquad b_3,\ \ldots,\ b_q,\ L+M\,; \end{matrix}\ \mu\right)$$

$Re(L) > 0,\ Re(M) > 0.$

(*i*) with $\alpha=\beta=0$, $L=\overline{a_1}$ and $M=\overline{b_2}-\overline{a_1}$ in (3·10), we have a known result ( [1], p. 810, equation (18) ).

(*ii*) Substituting $p = q = 3$, $a_2 = \frac{1}{2}$, $a_3 = \rho$, $b_3 = \sigma$ in (3·10), we obtain an integral involving a generalised Rice's polynomial (2·3),

$$(3{\cdot}11) \qquad \int_0^1 x^{L-1}\,(1-x)^{M-1}\, H_n^{(\alpha,\beta)}(\rho,\sigma,\mu x)\,dx$$

$$= \frac{(1+\alpha)_n\,\Gamma(L)\,\Gamma(M)}{\underline{|n}\;\Gamma(L+M)}\ {}_4F_3\left(\begin{matrix} -n,\ n+\alpha+\beta+1,\ \rho,\ L\,; \\ 1+\alpha,\ \sigma,\ L+M\,; \end{matrix}\ \mu\right)$$

$Re(L) > 0,\ Re(M) > 0.$

Further setting $L = \sigma$, and $M = \rho - \sigma$ in (3·11), we get a known result ( [2], p. 158).

Replacing $\alpha$, $\beta$ by $\frac{\bar{\alpha}+\bar{\beta}}{2}$ and $L = \frac{\bar{\alpha}+\bar{\beta}}{2} + 1$, $M = \frac{\bar{\alpha}-\bar{\beta}}{2}$, $\mu = \frac{y}{2}$ in (3·11), we obtain a known result ([7], p. 116, equation 7·4·7).

(*iii*) In (3·10), setting $p = q = 2$, $a_2 = \frac{1}{2}$, $\mu = 1$, $L = \sigma + 1$, $M = \beta + 1$, and with the help of Saalschutz's theorem, we have a known result ([8], p. 284).

(*iv*) Substituting $p = q = 2$, $a_2 = \frac{1}{2}$, $L = \xi$, $M = p - \xi$, in (3·10), we obtain a known result ([2], p. 157, equation (2·1)).

(*v*) In (3·10), with $p = 2$, $q = 3$, $a_2 = \frac{1}{2}$, $b_3 = 1$ and $\alpha = \beta = 0$, we get an integral involving a Bateman's polynomial,

$$\int_0^1 x^{L-1}\,(1-x)^{M-1}\, Z_n\,(\mu x)\,dx = \frac{\Gamma(L)\,\Gamma(M)}{\Gamma(L+M)}\ {}_3F_3\left(\begin{matrix} -n,\ n+1,\ L\,; \\ 1,\ 1,\ L+M\,; \end{matrix}\ \mu\right)$$

$Re(L) > 0,\ Re(M) > 0.$

(b) Setting $l = k = 3$, $\rho_2 = \frac{1}{2}$, $\rho_3 = \rho$, $\sigma_3 = \sigma$ in (3·9), we have an integral involving a product of generalised Sister Celine and generalised Rice's polynomials,

$$\int_0^1 x^{L-1} (1-x)^{M-1} f_n^{(\alpha,\beta)}\left(\begin{matrix} a_2, \ldots, a_p; \\ b_3, \ldots, b_q; \end{matrix} \mu x\right) H_m^{(\gamma,\delta)}(\rho, \sigma, \lambda x)\, dx$$

$$= \frac{\Gamma(L)\,\Gamma(M)\,(1+\alpha)_n\,(1+\gamma)_m}{\Gamma(L+M)\,\underline{|n}\,\underline{|m}} \sum_{r=0}^{\infty} \frac{(-n)_r\,(n+\alpha+\beta+1)_r \prod\limits_{j=2}^{p} (a_j)_r\,(L)_r\,\mu^r}{\underline{|r}\,(1+\alpha)_r\,(\frac{1}{2})_r \prod\limits_{j=3}^{q} (b_j)_r\,(L+M)_r}$$

$$\times {}_4F_3\left(\begin{matrix} -m, m+\gamma+\delta+1, \rho, L+r; \\ 1+\gamma, \sigma, L+M+r; \end{matrix} \lambda\right)$$

$Re(L) > 0$, $Re(M) > 0$.

Substituting $p = q = 3$, $a_2 = \frac{1}{2}$, $a_3 = \xi$, $b_3 = p$, we obtain an integral involving a product of two generalised Rice's polynomials,

$$\int_0^1 x^{L-1} (1-x)^{M-1} H_n^{(\alpha,\beta)}(\xi, p, \mu x)\, H_m^{(\gamma,\delta)}(\rho, \sigma, \lambda x)\, dx$$

$$= \frac{\Gamma(L)\,\Gamma(M)\,(1+\alpha)_n\,(1+\gamma)_m}{\Gamma(L+M)\,\underline{|n}\,\underline{|m}} \sum_{r=0}^{\infty} \frac{(-n)_r\,(n+\alpha+\beta+1)_r\,(\xi)_r\,(L)_r\,\mu^r}{\underline{|r}\,(1+\alpha)_r\,(p)_r\,(L+M)_r}$$

$$\times {}_4F_3\left(\begin{matrix} -m, m+\gamma+\delta+1, \rho, L+r; \\ 1+\gamma, \sigma, L+M+r; \end{matrix} \lambda\right)$$

$Re(L) > 0$, $Re(M) > 0$.

(c) When $l = 2$, $k = 3$, $\rho_2 = \frac{1}{2}$, $\sigma_3 = 1$, $\gamma = \delta = 0$ in (3·9), we have an integral involving a product of generalised Sister Celine and Bateman's polynomials,

$$\int_0^1 x^{L-1} (1-x)^{M-1} f_n^{(\alpha,\beta)}\left(\begin{matrix} a_2, \ldots, a_p; \\ b_3, \ldots, b_q; \end{matrix} \mu x\right) Z_m(\lambda x)\, dx$$

$$= \frac{\Gamma(L)\,\Gamma(M)\,(1+\alpha)_n}{\Gamma(L+M)\,\underline{|n}} \sum_{r=0}^{\infty} \frac{(-n)_r\,(n+\alpha+\beta+1)_r \prod\limits_{j=2}^{p} (a_j)_r\,(L)_r\,\mu^r}{\underline{|r}\,(1+\alpha)_r\,(\frac{1}{2})_r \prod\limits_{j=3}^{q} (b_j)_r\,(L+M)_r}$$

$$\times {}_3F_3\left(\begin{matrix} -m, m+1, L+r; \\ 1, 1, L+M+r; \end{matrix} \lambda\right)$$

$Re(L) > 0$, $Re(M) > 0$.

With $p = 2$, $q = 3$, $a_2 = \frac{1}{2}$, $b_3 = 1$, $\alpha = \beta = 0$, we obtain an integral involving a product of two Bateman's polynomials,

$$\int_0^1 x^{L-1} (1-x)^{M-1} Z_n(\mu x)\, Z_m(\lambda x)\, dx$$

$$= \frac{\Gamma(L)\,\Gamma(M)}{\Gamma(L+M)} \sum_{r=0}^{\infty} \frac{(-n)_r\,(n+1)_r\,(L)_r\,\mu^r}{\lfloor r\;(1)_r\,(1)_r\,(L+M)_r}\; {}_3F_3\left(\begin{matrix}-m,\, m+1,\, L+r\,;\\ 1,\, 1,\, L+M+r\,;\end{matrix}\;\lambda\right)$$

$Re(L) > 0$, $Re(M) > 0$.

(*d*) Setting $p = q = 2$, $a_2 = \frac{1}{2}$, $L = \sigma + 1$, $M = \beta + 1$, $\mu = 1$, in (3·9), we get an integral involving a product of Jacobi and generalised Sister Celine polynomials,

$$\int_0^1 x^{\sigma}\,(1-x)^{\beta}\,P_n^{(\alpha,\beta)}(1-2x)\; f_m^{(\gamma,\delta)}\left(\begin{matrix}\rho_2,\,\ldots,\,\rho_l\,;\\ \sigma_3,\,\ldots,\,\sigma_k\,;\end{matrix}\;\lambda x\right)dx$$

$$= \frac{\Gamma(\sigma+1)\,\Gamma(\beta+1)\,(1+\alpha)_n\,(1+\gamma)_m}{\Gamma(\sigma+\beta+2)\,\lfloor n\;\lfloor m} \sum_{r=0}^{\infty} \frac{(-n)_r\,(n+\alpha+\beta+1)_r\,(\sigma+1)_r}{\lfloor r\;(1+\alpha)_r\,(\sigma+\beta+2)_r}$$

$$\times\; {}_{l+2}F_{k+1}\left(\begin{matrix}-m,\, m+\gamma+\delta+1,\, \rho_2,\,\ldots,\,\rho_l,\, \sigma+r+1\,;\\ 1+\gamma,\, \frac{1}{2},\, \sigma_3,\,\ldots\ldots,\,\sigma_k,\, \sigma+\beta+r+2\,;\end{matrix}\;\lambda\right)$$

$Re(\sigma) > -1$, $Re(\beta) > -1$.

Expressing the hypergeometric function into the Contour integral of Barnes' type, interchanging the order of summation and integration and using Saalschutz's theorem, we have

$$\text{(3·12)}\qquad \int_0^1 x^{\sigma}\,(1-x)^{\beta}\,P_n^{(\alpha,\beta)}(1-2x)\; f_m^{(\gamma,\delta)}\left(\begin{matrix}\rho_2,\,\ldots,\,\rho_l\,;\\ \sigma_3,\,\ldots,\,\sigma_k\,;\end{matrix}\;\lambda x\right)dx$$

$$= \frac{(1+\gamma)_m\,\Gamma(1+\beta+n)\,\Gamma(\sigma+1)\,(\alpha-\sigma)_n}{\lfloor m\;\lfloor n\;\Gamma(\sigma+\beta+n+2)}$$

$$\times\; {}_{l+3}F_{k+2}\left(\begin{matrix}-m,\, m+\gamma+\delta+1,\, \rho_2,\,\ldots,\,\rho_l,\, 1+\sigma,\, 1-\alpha+\sigma\,;\\ 1+\gamma,\, \frac{1}{2},\, \sigma_3,\,\ldots,\,\sigma_k,\, 1-\alpha+\sigma-n,\, 2+\sigma+\beta+n\,;\end{matrix}\;\lambda\right)$$

$Re(\sigma) > -1$, $Re(\beta) > -1$.

(*i*) Substituting $l = k = 3$, $\rho_2 = \frac{1}{2}$, $\rho_3 = \xi$, $\sigma_3 = p$ in (3·12), we have

$$\text{(3·13)}\qquad \int_0^1 x^{\sigma}\,(1-x)^{\beta}\,P_n^{(\alpha,\beta)}(1-2x)\; H_m^{(\gamma,\delta)}(\xi,\, p,\, \lambda x)\,dx$$

$$= \frac{(1+\gamma)_m\,\Gamma(1+\beta+n)\,\Gamma(\sigma+1)\,\Gamma(\alpha-\sigma+n)}{\lfloor m\;\lfloor n\;\Gamma(\alpha-\sigma)\,\Gamma(\sigma+\beta+n+2)}$$

$$\times\; {}_5F_4\left(\begin{matrix}-m,\, m+\gamma+\delta+1,\, \xi,\, 1+\sigma,\, 1-\alpha+\sigma\,;\\ 1+\gamma,\, p,\, 1-\alpha+\sigma-n,\, \sigma+\beta+n+2\,;\end{matrix}\;\lambda\right)$$

$Re(\sigma) > -1$, $Re(\beta) > -1$,

an integral involving the product of Jacobi and generalised Rice's polynomials.

With $\xi = p$, $\lambda = 1$ in (3·13) we obtain a known result ([8], p. 288).

(*ii*) In (3·12), putting $l = 2$, $k = 3$, $\rho_2 = \frac{1}{2}$, $\sigma_3 = 1$, $\gamma = \delta = 0$, we have an integral involving the product of Jacobi and Bateman's polynomials,

3·3. (*a*) Setting $p = 0$, $q = 1$, $l = 0$, $k = 1$, $b_1 = 1 + \alpha$, $\sigma_1 = 1 + \beta$ in (3·7) and multiplying both sides by $\frac{(1+\alpha)_n\,(1+\beta)_m}{\lfloor n\;\lfloor m}$, we obtain an integral involving the product of two generalised Laguerre polynomials,

$$(3\cdot 14) \qquad \int_0^1 x^{L-1}(1-x)^{M-1} L_n^{(\alpha)}(\mu x) L_m^{(\beta)}(\lambda x)\, dx$$

$$= \frac{(1+\alpha)_n (1+\beta)_m \Gamma(L)\Gamma(M)}{n!\, m!\, \Gamma(L+M)} \sum_{r=0}^{\infty} \frac{(-n)_r (L)_r \mu^r}{r!\, (1+\alpha)_r (L+M)_r}$$

$$\times {}_2F_2\left(\begin{matrix} -m, L+r\,; \\ 1+\beta, L+M+r\,; \end{matrix} \lambda\right)$$

$Re(L) > 0$, $Re(M) > 0$.

When $m = 0$, (3·14) reduces to an integral involving a generalised Laguerre polynomial,

$$\int_0^1 x^{L-1}(1-x)^{M-1} L_n^{(\alpha)}(\mu x)\, dx = \frac{(1+\alpha)_n \Gamma(L)\Gamma(M)}{n!\, \Gamma(L+M)}\, {}_2F_2\left(\begin{matrix} -n, L\,; \\ 1+\alpha, L+M\,; \end{matrix} \mu\right)$$

$Re(L) > 0$, $Re(M) > 0$,

which further reduces to a known result ([8], p. 293 (5) on putting $L = 1+\alpha$ and $M = \beta - \alpha$.

(b) In (3·7), setting $p = q = 1$, $a_1 = n+\alpha+\beta+1$, $b_1 = 1+\alpha$, $\mu = 1$ $l = 0$, $k = 1$, $\sigma_1 = 1+\gamma$, $L = \sigma + 1$, $M = \beta + 1$

and multiplying both sides by $\frac{(1+\alpha)_n (1+\gamma)_m}{n!\, m!}$, we get an integral involving the product of Jacobi and generalised Laguerre polynomials,

$$\int_0^1 x^{\sigma}(1-x)^{\beta} P_n^{(\alpha,\beta)}(1-2x) L_m^{(\gamma)}(\lambda x)\, dx$$

$$= \frac{(1+\alpha)_n (1+\gamma)_m \Gamma(\sigma+1)\Gamma(\beta+1)}{n!\, m!\, \Gamma(\sigma+\beta+2)} \sum_{r=0}^{\infty} \frac{(-n)_r (n+\alpha+\beta+1)_r (\sigma+1)_r}{r!\, (1+\alpha)_r (\sigma+\beta+2)_r}$$

$$\times {}_2F_2\left(\begin{matrix} -m, \sigma+r+1\,; \\ 1+\gamma, \sigma+\beta+r+2\,; \end{matrix} \lambda\right)$$

$Re(\sigma) > -1$, $Re(\beta) > -1$.

Expressing the hypergeometric function into the Contour integral of Barnes' type, interchanging the order of summation and integration and using Saalschutz' theorem etc., we obtain

$$\int_0^1 x^{\sigma}(1-x)^{\beta} P_n^{(\alpha,\beta)}(1-2x) L_m^{(\gamma)}(\lambda x)\, dx$$

$$= \frac{(1+\gamma)_m \Gamma(1+\beta+n)\Gamma(\sigma+1)\Gamma(\alpha-\sigma+n)}{n!\, m!\, \Gamma(\alpha-\sigma)\Gamma(\sigma+\beta+n+2)}$$

$$\times {}_3F_3\left(\begin{matrix} -m, \sigma+1, 1-\alpha+\sigma\,; \\ 1+\gamma, 1-\alpha+\sigma-n, \sigma+\beta+n+2\,; \end{matrix} \lambda\right)$$

$Re(\sigma) > -1$, $Re(\beta) > -1$.

3·4. In (3·3), substituting $\delta = \gamma = 2$, $c = d = -2$, we obtain

$$(3\cdot15) \qquad \int_0^1 x^{L+n+m-1} (1-x)^{M-1} {}_{p+2}F_q \left[ \begin{matrix} \frac{-n}{2}, \frac{-n+1}{2}, a_p; \\ b_q; \end{matrix} \mu x^{-2} \right] \times {}_{l+2}F_k \left[ \begin{matrix} \frac{-m}{2}, \frac{-m+1}{2}, \rho_l; \\ \sigma_k; \end{matrix} \lambda x^{-2} \right] dx$$

$$= \Gamma(M) \sum_{r=0}^{\infty} \sum_{s=0}^{r} \frac{\left(\frac{-n}{2}\right)_r \left(\frac{-n+1}{2}\right)_r (a_p)_r \mu^r \left(\frac{-m}{2}\right)_s \left(\frac{-m+1}{2}\right)_s (\rho_l)_s (1-b_q-r)_s (-r)_s}{\lfloor r \lfloor s (b_q)_r (\sigma_k)_s \left(1+\frac{n}{2}-r\right)_s \left(1+\frac{n-1}{2}-r\right)_s (1-a_p-r)_s}$$

$$\times \left(\frac{\lambda}{\mu}\right)^s (-1)^{(p+q+1)s} \frac{\Gamma(n+m+L-2r)}{\Gamma(n+m+L+M-2r)}$$

$Re(L)+n+m > 0$, $Re(M) > 0$.

Replacing $r$ by $r+s$, using relations $\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-1)^n}{(\alpha)_n}$, $(\alpha)_{2n} = 2^{2n} (\alpha/2)_n \left(\frac{\alpha+1}{2}\right)_n$,

$(\alpha)_{n+k} = (\alpha)_k (\alpha+k)_n$ and $\frac{(A+r)_s}{(1-A-r-s)} = (-1)^s$, in (3·15), we obtain the integral

$$(3\cdot16) \qquad \int_0^1 x^{L+n+m-1} (1-x)^{M-1} {}_{p+2}F_q \left[ \begin{matrix} \frac{-n}{2}, \frac{-n+1}{2}, a_p; \\ b_q; \end{matrix} \mu x^{-2} \right] \times {}_{l+2}F_k \left[ \begin{matrix} \frac{-m}{2}, \frac{-m+1}{2}, \rho_l; \\ \sigma_k; \end{matrix} \lambda x^{-2} \right] dx$$

$$= \frac{\Gamma(M)\, \Gamma(L+m+n)}{\Gamma(L+M+m+n)} \sum_{r=0}^{\infty} \frac{\left(\frac{-n}{2}\right)_r \left(\frac{-n+1}{2}\right)_r (a_p)_r \left(\frac{1-L-M-n-m}{2}\right)_r \left(\frac{2-L-M-n-m}{2}\right)_r \mu^r}{\lfloor r\, (b_q)_r \left(\frac{1-L-m-n}{2}\right)_r \left(\frac{2-L-m-n}{2}\right)_r}$$

$$\times {}_{l+4}F_{k+2} \left( \begin{matrix} \frac{-m}{2}, \frac{-m+1}{2}, \rho_l, \frac{1-L-M-n-m}{2}+r, \frac{2-L-M-n-m}{2}+r; \\ \sigma_k, \frac{1-L-m-n}{2}+r, \frac{2-L-m-n}{2}+r \quad ; \end{matrix} \lambda \right)$$

$Re(L)+n+m > 0$, $Re(M) > 0$.

**Special Cases :**

(*a*) In (3·16), setting $p = 1$, $q = 2$, $a_1 = \gamma - \beta$, $b_1 = \gamma$, $b_2 = 1-\beta-n$, $\mu = 1$, $l = 1$, $k = 2$, $\rho_1 = y-B$, $\sigma_1 = y$, $\sigma_2 = 1-B-m$, $\lambda = 1$

and multiplying both sides by $\frac{2^{n+m} (\beta)_n (B)_m}{\lfloor n \lfloor m}$, we have

$$\int_0^1 x^{L-1}(1-x)^{M-1} R_n(\beta, \gamma ; x) R_m(B, y ; x)\, dx$$

$$= \frac{\Gamma(M)\,\Gamma(L+m+n)\, 2^{n+m}\,(\beta)_n\,(B)_m}{\Gamma(L+M+m+n)\, \underline{|n}\, \underline{|m}}$$

$$\sum_{r=0}^{\infty} \frac{\left(\frac{-n}{2}\right)_r \left(\frac{-n+1}{2}\right)_r (\gamma-\beta)_r \left(\frac{1-L-M-n-m}{2}\right)_r \left(\frac{2-L-M-n-m}{2}\right)_r}{\underline{|r}\,(\gamma)_r\,(1-\beta-n)_r \left(\frac{1-L-m-n}{2}\right)_r \left(\frac{2-L-m-n}{2}\right)_r}$$

$$\times\, {}_5F_4\left(\begin{matrix} \frac{-m}{2}, \frac{-m+1}{2}, y-B, \frac{1-L-M-n-m}{2}+r, \frac{2-L-M-n-m}{2}+r ; \\ y, 1-B-m, \frac{1-L-m-n}{2}+r, \frac{2-L-m-n}{2}+r ; \end{matrix} \; 1\right)$$

$Re(M) > 0$, $Re(L) > 0$,

an integral involving the product of two Bedient's polynomials.

When $m = 0$, we obtain an integral involving the Bedient's polynomial,

$$\int_0^1 x^{L-1}(1-x)^{M-1} R_n(\beta, \gamma ; x)\, dx$$

$$= \frac{\Gamma(M)\,\Gamma(L+n)\, 2^n\,(\beta)_n}{\Gamma(L+M+n)\, \underline{|n}}\; {}_5F_4\left(\begin{matrix} \frac{-n}{2}, \frac{-n+1}{2}, \gamma-\beta, \frac{1-L-M-n}{2}, \frac{2-L-M-n}{2} ; \\ \gamma, 1-\beta-n, \frac{1-L-n}{2}, \frac{2-L-n}{2} ; \end{matrix} \; 1\right)$$

$Re(L) > 0$, $Re(M) > 0$.

(*b*) In (3·16), substituting $p = q = 0$, $l = k = 0$, $\lambda = \mu = -1$ and multiplying both sides by $2^{n+m}$, we obtain an integral involving a product of two Hermite polynomials,

$$\int_0^1 x^{L-1}(1-x)^{M-1} H_n(x)\, H_m(x)\, dx$$

$$= \frac{2^{m+n}\,\Gamma(M)\,\Gamma(L+m+n)}{\Gamma(L+M+m+n)} \sum_{r=0}^{\infty} \frac{\left(\frac{-n}{2}\right)_r \left(\frac{-n+1}{2}\right)_r \left(\frac{1-L-M-n-m}{2}\right)_r \left(\frac{2-L-M-n-m}{2}\right)_r (-1)^r}{\underline{|r} \left(\frac{1-L-m-n}{2}\right)_r \left(\frac{2-L-m-n}{2}\right)_r}$$

$$\times\, {}_4F_2\left(\begin{matrix} \frac{-m}{2}, \frac{-m+1}{2}, \frac{1-L-M-n-m}{2}+r, \frac{2-L-M-n-m}{2}+r ; \\ \frac{1-L-m-n}{2}+r, \frac{2-L-m-n}{2}+r ; \end{matrix} \; -1\right)$$

$Re(L) > 0$, $Re(M) > 0$.

When $m = 0$, we have an integral involving a Hermite polynomial,

$$\int_0^1 x^{L-1} (1-x)^{M-1} H_n(x)\, dx$$

$$= \frac{2^n \Gamma(M) \Gamma(L+n)}{\Gamma(L+M+n)} {}_4F_3\left( \begin{matrix} \frac{-n}{2}, \frac{-n+1}{2}, \frac{1-L-M-n}{2}, \frac{2-L-M-n}{2}; \\ \frac{1-L-n}{2}, \frac{2-L-n}{2} \quad ; \end{matrix} -1 \right)$$

$Re(L) > 0$, $Re(M) > 0$.

**4.** Now we have, from (3·1), an infinite integral involving the product of two generalised hypergeometric polynomials and an exponential factor.

$$(4\cdot1) \qquad \int_0^\infty e^{-x}\, x^{L+(\delta-1)n+(\gamma-1)m-1}\, {}_{p+\delta}F_q\left[ \begin{matrix} \triangle(\delta,-n),\, a_p\,; \\ b_q\,; \end{matrix}\, \mu x^c \right]$$

$$\times\, {}_{l+\gamma}F_k \left[ \begin{matrix} \triangle(\gamma,-m),\, \rho_l\,; \\ \sigma_k\,; \end{matrix}\, \lambda x^d \right] dx$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{r} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-n+i}{\delta}\right)_r \prod_{i=0}^{\gamma-1} \left(\frac{-m+i}{\gamma}\right)_s (a_p)_r\, (1-b_q-r)_s\, (\rho_l)_s\, (-r)_s\, \mu^r}{\lfloor r \;\lfloor s \; \prod_{i=0}^{\delta-1} \left(1+\frac{n-i}{\delta}-r\right)_s (b_q)_r\, (1-a_p-r)_s\, (\sigma_k)_s}$$

$$\times \left(\frac{\lambda}{\mu}\right)^s (-1)^{(p+q+\delta+1)s}\, \Gamma\{L+(\delta-1)n+(\gamma-1)m+c(r-s)+ds\}$$

$Re(L) > (1-\delta)n+(1-\gamma)m$.

In (4·1), substituting $\delta = \gamma = 1$, $c = d = 1$, replacing $r$ by $r+s$ and using the relations $(\alpha)_{n+k} = (\alpha)_k\, (\alpha+k)_n$, $\frac{(A+r)_s}{(1-A-r-s)_s} = (-1)^s$; we have

$$(4\cdot2) \qquad \int_0^\infty e^{-x}\, x^{L-1}\, {}_{p+1}F_q\left[ \begin{matrix} -n,\, a_p\,; \\ b_q; \end{matrix}\, \mu x \right] \; {}_{l+1}F_k\left[ \begin{matrix} -m,\, \rho_l\,; \\ \sigma_k; \end{matrix}\, \lambda x \right] dx$$

$$= \Gamma(L) \sum_{r=0}^{\infty} \frac{(-n)_r\, (a_p)_r\, (L)_r\, \mu^r}{\lfloor r \; (b_q)_r}\, {}_{l+2}F_k\left( \begin{matrix} -m,\, \rho_l,\, L+r\,; \\ \sigma_k\,; \end{matrix}\, \lambda \right)$$

$Re(L) > 0$.

**4·1.** In (4·2), setting $a_1 = n+\alpha+\beta+1$, $b_1 = 1+\alpha$, $b_2 = \frac{1}{2}$,
$\rho_1 = m+\gamma+\delta+1$, $\sigma_1 = 1+\gamma$, $\sigma_2 = \frac{1}{2}$

and multiplying both sides by $\frac{(1+\alpha)_n\, (1+\gamma)_m}{\lfloor n \;\lfloor m}$, we obtain an integral involving the product of two generalised Sister Celine's polynomials,

$$(4\cdot3) \qquad \int_0^\infty e^{-x}\, x^{L-1}\, f_n^{(\alpha,\beta)}\left( \begin{matrix} a_2, \ldots, a_p\,; \\ b_3, \ldots, b_q\,; \end{matrix}\, \mu x \right) f_m^{(\gamma,\delta)}\left( \begin{matrix} \rho_2, \ldots, \rho_l\,; \\ \sigma_3, \ldots, \sigma_k\,; \end{matrix}\, \lambda x \right) dx$$

$$= \Gamma(L) \frac{(1+\alpha)_n (1+\gamma)_m}{\lfloor n \lfloor m} \sum_{r=0}^{\infty} \frac{(-n)_r (n+\alpha+\beta+1)_r \prod_{j=2}^{p} (a_j)_r (L)_r \mu^r}{\lfloor r (1+\alpha)_r (\frac{1}{2})_r \prod_{j=3}^{q} (b_j)_r}$$

$$\times {}_{l+2}F_k\left(\begin{matrix}-m, m+\gamma+\delta+1, \rho_2, \ldots, \rho_l, L+r ; \\ 1+\gamma, \frac{1}{2}, \sigma_3, \ldots, \sigma_k \quad ;\end{matrix} \lambda\right)$$

$Re(L) > 0$.

(*a*) When $m = 0$, we have an integral involving a generalised Sister Celine's polynomial,

$$(4{\cdot}4) \qquad \int_0^{\infty} e^{-x} x^{L-1} f_n^{(\alpha,\beta)}\left(\begin{matrix}a_2, \ldots, a_p ; \\ b_3, \ldots, b_q ;\end{matrix} \mu x\right) dx$$

$$= \Gamma(L) \frac{(1+\alpha)_n}{\lfloor n} {}_{p+2}F_q \left(\begin{matrix}-n, n+\alpha+\beta+1, a_2, \ldots, a_p, L ; \\ 1+\alpha, \frac{1}{2}, b_3, \ldots, b_q \quad ;\end{matrix} \mu\right)$$

$Re(L) > 0$.

(*i*) In (4·4), putting $\alpha = \beta = 0$, $b_3 = \frac{1}{2}$ and $L = \frac{1}{2}$, we obtain a known result ([1], p. 810, eqn. (17) ).

(*ii*) With $p = 2$, $q = 3$, $a_2 = \frac{1}{2}$, $b_3 = p$, $L = \xi$, (4·4) yields a known result ( [2], p. 158, eqn. (3·3) ).

Substituting in (4·4), $p = q = 3$, $a_2 = \frac{1}{2}$, $a_3 = \xi$, $b_3 = p$, we obtain an integral involving a generalised Rice's polynomial,

$$(4{\cdot}5) \qquad \int_0^{\infty} e^{-x} x^{L-1} H_n^{(\alpha,\beta)}(\xi, p, \mu x) dx$$

$$= \Gamma(L) \frac{(1+\alpha)_n}{\lfloor n} {}_4F_2 \left(\begin{matrix}-n, n+\alpha+\beta+1, \xi, L ; \\ 1+\alpha, \qquad p ;\end{matrix} \mu\right)$$

$Re(L) > 0$.

With $\xi = p$, in (4·5), we obtain an integral containing a Jacobi polynomial,

$$(4{\cdot}6) \qquad \int_0^{\infty} e^{-x} x^{L-1} P_n^{(\alpha,\beta)} (1-2\mu x) dx$$

$$= \frac{\Gamma(L) \Gamma(1+\alpha)_n}{\lfloor n} {}_3F_1\left(\begin{matrix}-n, n+\alpha+\beta+1, L ; \\ 1+\alpha \quad ;\end{matrix} \mu\right)$$

$Re(L) > 0$.

Setting $p = 2$, $q = 3$, $a_2 = \frac{1}{2}$, $b_3 = 1$, $\alpha = \beta = 0$, in (4·4), we get an integral involving a Bateman's polynomial,

$$(4\cdot 7)\quad \int_0^\infty e^{-x}\, x^{L-1}\, Z_n(\mu x)\, dx = \Gamma(L)\, {}_3F_2\left(\begin{matrix} -n,\ n+1,\ L\ ; \\ 1,\quad 1\quad ; \end{matrix}\ \mu\right)\quad Re\,(L) > 0.$$

(*b*) Substituting $l = k = 3$, $\rho_2 = \frac{1}{2}$, $\rho_3 = \xi$, $\sigma_3 = p$ in (4·3), we obtain an integral involving a product of generalised Sister Celine's and generalised Rice's polynomials,

$$(4\cdot 8)\qquad \int_0^\infty e^{-x}\, x^{L-1}\, f_n^{(\alpha,\,\beta)}\left(\begin{matrix} a_2, \ldots\ldots, a_p\ ; \\ b_3, \ldots\ldots, b_q\ ; \end{matrix}\ \mu x\right) H_m^{(\gamma,\,\delta)}(\xi,\, p,\, \lambda x)\, dx$$

$$= \Gamma(L)\, \frac{(1+\alpha)_n\,(1+\gamma)_m}{\underline{|n}\ \underline{|m}} \sum_{r=0}^{\infty} \frac{(-n)_r\,(n+\alpha+\beta+1)_r \prod_{j=2}^{p} (a_j)_r\, (L)_r\, \mu^r}{\underline{|r}\ (1+\alpha)_r\, (\frac{1}{2})_r \prod_{j=3}^{q} (b_j)_r}$$

$${}_4F_2\left(\begin{matrix} -m,\ m+\gamma+\delta+1,\ \xi,\ L+r\ ; \\ 1+\gamma,\qquad p\qquad\qquad ; \end{matrix}\ \lambda\right)\qquad Re\ (L) > 0.$$

With $p = q = 3$, $a_2 = \frac{1}{2}$, $a_3 = \rho$, $b_3 = \sigma$ in (4·8), we have an integral involving a product of two generalised Rice's polynomials,

$$(4\cdot 9)\qquad \int_0^\infty e^{-x}\, x^{L-1}\, H_n^{(\alpha,\beta)}(\rho,\, \sigma,\, \mu x)\; H_m^{(\gamma,\delta)}(\xi,\, p,\, \lambda x)\, dx$$

$$= \Gamma(L)\, \frac{(1+\alpha)_n\,(1+\gamma)_m}{\underline{|n}\ \underline{|m}} \sum_{r=0}^{\infty} \frac{(-n)_r\,(n+\alpha+\beta+1)_r\,(\rho)_r\,(L)_r\,\mu^r}{\underline{|r}\ (1+\alpha)_r\,(\sigma)_r}$$

$$\times\ {}_4F_2\left(\begin{matrix} -m,\ m+\gamma+\delta+1,\ \xi,\ L+r\ ; \\ 1+\gamma,\qquad p\qquad\qquad , \end{matrix}\ \lambda\right)\qquad Re\,(L) > 0.$$

Substituting $\rho = \sigma$ and $\xi = p$ in (4·9,) we obtain an integral involving a product of Jacobi polynomials,

$$\int_0^\infty e^{-x}\, x^{L-1}\, P_n^{(\alpha,\beta)}(1 - 2\,\mu x)\; P_m^{(\gamma,\delta)}(1 - 2\,\lambda x)\, dx$$

$$= \Gamma(L)\, \frac{(1+\alpha)_n\,(1+\gamma)_m}{\underline{|n}\ \underline{|m}} \sum_{r=0}^{\infty} \frac{(-n)_r\,(n+\alpha+\beta+1)_r\,(L)_r\,\mu^r}{\underline{|r}\ (1+\alpha)_r}$$

$$\times\ {}_3F_1\left(\begin{matrix} -m,\ m+\gamma+\delta+1,\ L+r\ ; \\ 1+\gamma\qquad\qquad\qquad ; \end{matrix}\ \lambda\right)\qquad Re\,(L) > 0.$$

(*c*) When $l = 2$, $k = 3$, $\rho_2 = \frac{1}{2}$, $\sigma_3 = 1$, $\gamma = \delta = 0$ in (4·3), we get an integral involving a product of generalised Sister Celine and Bateman's polynomials,

$$(4\cdot 10)\qquad \int_0^\infty e^{-x}\, x^{L-1}\, f_n^{(\alpha,\beta)}\left(\begin{matrix} a_2, \ldots\ldots, a_p\ ; \\ b_3, \ldots\ldots, b_q\ ; \end{matrix}\ \mu x\right) Z_m(\lambda x)\, dx$$

$$= \Gamma(L) \frac{(1+\alpha)_n}{\lfloor n} \sum_{r=0}^{\infty} \frac{(-n)_r (n+\alpha+\beta+1)_r \prod_{j=2}^{p} (a_j)_r (L)_r \mu^r}{\lfloor r (1+\alpha)_r (\frac{1}{2})_r \prod_{j=3}^{q} (b_j)_r}$$

$$\times {}_3F_2 \left( \begin{matrix} -m, m+1, L+r ; \\ 1, \ 1 \qquad ; \end{matrix} \lambda \right) \quad Re\,(L) > 0.$$

Taking $p = 2, q=3, a_2 = \frac{1}{2}, b_3 = 1$ and $\alpha = \beta = 0$ in (4·10), we obtain an integral involving a product of two Bateman's polynomials,

$$\int_0^{\infty} e^{-x} x^{L-1} Z_n(\mu x) Z_m(\lambda x)\, dx = \Gamma(L) \sum_{r=0}^{\infty} \frac{(-n)_r (n+1)_r (L)_r \mu^r}{\lfloor r (1)_r (1)_r}$$

$$\times {}_3F_2 \left( \begin{matrix} -m, m+1, L+r ; \\ 1, \ 1 \qquad ; \end{matrix} \lambda \right) \quad Re\,(L) > 0.$$

4·2. Setting $p = 0, q = 1, b_1 = 1 + \alpha$,

$l = 0, k = 1, \sigma_1 = 1 + \beta$ in (4·2) and multiplying both sides by $\frac{(1+\alpha)_n (1+\beta)_m}{\lfloor n \ \lfloor m}$, we obtain an integral involving a product of two generalised Laguerre polynomials,

(4·11)
$$\int_0^{\infty} e^{-x} x^{L-1} L_n^{(\alpha)}(\mu x) L_m^{(\beta)}(\lambda x)\, dx$$

$$= \Gamma(L) \frac{(1+\alpha)_n (1+\beta)_m}{\lfloor n \ \lfloor m} \sum_{r=0}^{\infty} \frac{(-n)_r (L)_r \mu^r}{\lfloor r (1+\alpha)_r} {}_2F_1 \left( \begin{matrix} -m, L+r ; \\ 1+\beta ; \end{matrix} \lambda \right) Re\,(L) > 0$$

With $\mu = \lambda = 1, L = \alpha + \beta + 1$ in (4·11), we obtain a known result ([8], p. 293(4) ), on using Gauss's theorem and relation $\Gamma(z)\,\Gamma(1-z) = \pi/\sin \pi z$.

When $m = 0$, (4·11) reduces to an integral involving a generalised Laguerre polynomial,

$$\int_0^{\infty} e^{-x} x^{L-1} L_n^{(\alpha)}(\mu x)\, dx = \Gamma(L) \frac{(1+\alpha)}{\lfloor n} {}_2F_1 \left( \begin{matrix} -n, L ; \\ 1+\alpha ; \end{matrix} \mu \right) \quad Re\,(L) > 0,$$

which further reduces to a known result ( [8], p. 292(1) ), when $L = \beta$, $\mu = 1$ and using Gauss's theorem.

4·3. In (4·1), substituting $\delta = \gamma = 2, c = d = -2$, replacing $r$ by $r + s$, using relations $\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-1)^n}{(\alpha)_n}$, $(\alpha)_{2n} = 2^{2n} \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n$, $(\alpha)_k (\alpha+k)_n = (\alpha)_{n+k}$ and $\frac{(A+r)_s}{(1-A-r-s)_s} = (-1)^s$, we obtain the integral

$$(4{\cdot}12)\quad \int_0^\infty e^{-x}\, x^{L+n+m-1}\; {}_{p+2}F_q\left[\begin{matrix}\frac{-n}{2}, \frac{-n+1}{2}, a_p ; \mu\, x^{-2}\\ b_q ;\end{matrix}\right] {}_{l+2}F_k\left[\begin{matrix}\frac{-m}{2}, \frac{-m+1}{2}, \rho_l ; \lambda x^{-2}\\ \sigma k ;\end{matrix}\right] dx$$

$$= \Gamma(L+m+n) \sum_{r=0}^{\infty} \frac{\left(\frac{-n}{2}\right)_r \left(\frac{-n+1}{2}\right)_r (a_p)_r}{\lfloor r \; (b_q)_r \left(\frac{1-L-m-n}{2}\right)_r \left(\frac{2-L-m-n}{2}\right)_r} \left(\frac{\mu}{2^2}\right)^r$$

$$\times\; {}_{l+2}F_{k+2}\left(\begin{matrix}\frac{-m}{2}, \frac{-m+1}{2}, \rho_l & ; \\ \sigma_k, \frac{1-L-m-n}{2}+r, \frac{2-L-m-n}{2}+r & ; \frac{\lambda}{2^2}\end{matrix}\right)$$

$Re\,(L) > -n-m.$

(*a*) In (4·12), setting $p = 1$, $q = 2$, $a_1 = \gamma - \beta$, $b_1 = \gamma$, $b_2 = 1 - \beta - n$, $\mu = 1$, $l = 1$, $k = 2$, $\rho_1 = y - B$, $\sigma_1 = y$, $\sigma_2 = 1 - B - m$, $\lambda = 1$,

and multiplying both sides by $\dfrac{2^{n+m}\,(\beta)_n\,(B)_m}{\lfloor n \; \lfloor m}$, we have

$$\int_0^\infty e^{-x}\, x^{L-1}\, R_n(\beta, \gamma ; x)\, R_m(B, y ; x)\, dx$$

$$=\Gamma(L+m+n)\, \frac{2^{n+m}\,(\beta)_n\,(B)_m}{\lfloor n \; \lfloor m} \sum_{r=0}^{\infty} \frac{\left(\frac{-n}{2}\right)_r \left(\frac{-n+1}{2}\right)_r (\gamma-\beta)_r}{\lfloor r\, (\gamma)_r\, (1-\beta-n)_r \left(\frac{1-L-m-n}{2}\right)_r \left(\frac{2-L-m-n}{2}\right)_r} \left(\frac{1}{2^2}\right)^r$$

$$\times\; {}_3F_4\left(\begin{matrix}\frac{-m}{2}, \frac{-m+1}{2}, y-B & ; \\ y, 1-B-m, \frac{1-L-m-n}{2}+r, \frac{2-L-m-n}{2}+r & ; \frac{1}{2^2}\end{matrix}\right) \quad Re(L) > 0$$

an integral involving a product of two Bedient's polynomials.

When $m = 0$, we obtain an integral involving a Bedient's polynomial,

$$\int_0^\infty e^{-x}\, x^{L-1}\, R_n(\beta, \gamma ; x)\, dx$$

$$= \Gamma(L+n)\, \frac{2^n\,(\beta)_n}{\lfloor n}\; {}_3F_4\left(\begin{matrix}\frac{-n}{2}, \frac{-n+1}{2}, \gamma-\beta & ; \\ \gamma, 1-\beta-n, \frac{1-L-n}{2}, \frac{2-L-n}{2} & ; \frac{1}{2^2}\end{matrix}\right) Re\,(L) > 0.$$

(*b*) In (4·12), substituting $p = q = 0$, $l = k = 0$, $\mu = \lambda = -1$ and multiplying both sides by $2^{n+m}$, we have an integral involving the product of two Hermite polynomials,

$$\int_0^\infty e^{-x}\, x^{L-1}\, H_n(x)\, H_m(x)\, dx$$

$$= 2^{n+m}\,\Gamma(L+m+n) \sum_{r=0}^{\infty} \frac{\left(\frac{-n}{2}\right)_r \left(\frac{-n+1}{2}\right)_r}{\lfloor r \left(\frac{1-L-m-n}{2}\right)_r \left(\frac{2-L-m-n}{2}\right)_r} \left(-\frac{1}{2^2}\right)^r$$

$$\times\, {}_2F_2 \left( \begin{matrix} \frac{-m}{2}, \frac{-m+1}{2}, & ; & -\frac{1}{2^2} \\ \frac{1-L-m-n}{2}+r, \frac{2-L-m-n}{2}+r; & & \end{matrix} \right) \quad Re\,(L) > 0$$

When $m = 0$, we have an integral involving a Hermite polynomial,

$$\int_0^{\infty} e^{-x}\, x^{L-1}\, H_n(x)\, dx$$

$$= 2^n\,\Gamma(L+n)\; {}_2F_2 \left( \begin{matrix} \frac{-n}{2}, \frac{-n+1}{2} & ; & -\frac{1}{2^2} \\ \frac{1-L-n}{2}, \frac{2-L-n}{2} & ; & \end{matrix} \right) \quad Re\,(L) > 0.$$

**Acknowledgements**

I wish to express my sincere thanks to Dr. V. M. Bhise, G. S. Technological Institute, Indore for his help and guidance during the preparation of this paper. I am also thankful to the referee for some useful suggestions. My thanks are also due to Dr. S. M. Dasgupta, Principal, G. S. Technological Institute for the facilities he gave to me.

**References**

1. Fasenmyer, Sister M. Celine. Some generalised hypergeometric polynomials. *Bull. Math. Soc.*, **53** : 806–812, (1947).
2. Khandekar, P. R. On a generalisation of Rice's polynomial—1. *Proc. of the National Academy of Sciences, India, A*, **34** (2) : 157–162, (1964).
3. Rainville, E. D. Generating functions for Bessel and related polynomials, *Canadian Journal of Math.*, **5** : 104–106, (1953).
4. Bedient, P. E. Polynomials related to Apell functions of two Variables. *Michigan thesis*, (1958).
5. Erdelyi, A. *et al.* Higher transcendental functions. *McGraw Hill, New York.* **Vol I**, (1953.)
6. Saxena, R. K. On some results involving Jacobi polynomials, *Journal of the Indian Mathematical Society*, **Vol. 28**, : Nos. 3 & 4, (1964).
7. Khandekar, P. R. *Ph.D. Thesis, Vikram University, Ujjain*, (1964).
8. Erdelyi, A. *et al.* Tables of integral transfroms. *McGraw-Hill, New York*, **Vol. 2** : (1954).

# Relation between Whittaker and generalised Hankel transforms

By

O. P. SHARMA

*Department of Mathematics, Holkar Science College, Indore (M. P.)*

[Received on 14th September, 1966]

**Abstract**

In this paper a new generalisation of the Hankel transform has been introduced by using the Fourier kernel, given by Fox. A relation has been established between Whittaker transform of $t^{\mu} f(t)$ and the new generalised Hankel transform of $f(t)$. Certain new infinite integrals have been evaluated with the help of this relation.

## 1. Introduction

The Whittaker transform, introduced by Verma (18, p. 17) of a function $f(t) \in L(0, \infty)$ is

(1·1) $\phi(s : \lambda, \delta) = s\int_0^\infty (2\,st)^{-\frac{1}{2}}\, W_{\lambda, \delta}(2\,st) f(t)\, dt$, $Re(s) > 0$, which has been given a symbolic notation by Bose (4, p. 9) and denoted by

$$\phi(s : \lambda, \delta) \overset{\lambda}{\underset{\delta}{=}} f(t).$$

For $\lambda = \frac{1}{4}$, $\delta = \pm \frac{1}{4}$, (1·1) reduces to the classical Laplace transform

$$\phi(s) = s\int_0^\infty e^{-st} f(t)\, dt, \tag{1·2}$$

which will be denoted by

$$\phi(s) \doteqdot f(t).$$

Further Meijer (10, p. 599) introduced another generalisation of (1·2), as:

$$\phi(s : \nu) = s\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty (st)^{\frac{1}{2}} K_\nu(st) f(t)\, dt. \tag{1·3}$$

Rathie (14, p. 173) represented the K-transform (1·3) by

$$\phi(s : \nu) \overset{K}{\underset{\nu}{=}} f(t).$$

With $\nu = \pm \frac{1}{2}$, (1·3) reduces to (1·2).

The Hankel transform in Tricomi's form has been defined as :

$$(1{\cdot}4) \qquad g(\xi) = \int_0^\infty J_\nu\,(2\sqrt{\xi t}\,)\; h(t)\; dt.$$

A new generalisation of (1·4) may be introduced by using a Fourier kernel in terms of H-function, given by Fox (7, p. 408), by an integral equation :

$$(1{\cdot}5) \quad g(\xi) = \int_0^\infty \xi^{-1/4\gamma}\; t^{3/4\gamma-1}\; H\,{}^{q,\,p}_{2p,\,2q}\left[\xi t \,\middle|\, \begin{matrix} \{(a_p + \frac{2\gamma-1}{4\gamma}\alpha_p,\, \alpha_p)\}, \\ \{(b_q + \frac{2\gamma-1}{4\gamma}\beta_q,\, \beta_q)\}, \end{matrix} \right.$$

$$\left. \times \begin{matrix} \{(1 - a_p - \frac{2\gamma+1}{4\gamma}\alpha_p,\, \alpha_p)\} \\ \{(1 - b_q - \frac{2\gamma+1}{4\gamma}\beta_q,\, \beta_q)\} \end{matrix} \right] f(t)\; dt$$

provided that $\gamma$ is a real constant,

where

$$\frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\;\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\;\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\; . \; x^s\, ds$$

has been a symbolic notation by Gupta and Jain (9) as :

$$H^{m,\,n}_{p,\,q}\left[x \,\middle|\, \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix}\right],$$

which may further be represented in a more compact form as :

$$H^{m,\,n}_{p,\,q}\left[x \,\middle|\, \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix}\right],$$

where $\{(a_r, \alpha_r)\}$ stands for the set of parameters $(a_1, \alpha_1)$ , . . . , $(a_r, \alpha_r)$.

We shall denote (1·5) symbolically by

$$g(\xi) = H[f(t)\;;\;\xi\;;\;(a_p, \alpha_p)\;;\;(b_q, \beta_q)].$$

The object of this paper is to establish a relation between Whittaker transform of $t^\mu f(t)$, where $Re(\mu) > -1$, and the generalised Hankel transform of $f(t)$,

given by (1·5). This relation has been utilised for evaluating certain infinite integrals, which are believed to be new.

In section 2 some of the known properties of the H-function have been given, which will be used in our discussion and in section 3, we have evaluated an infinite integral, which will help us to establish the main result of this paper.

**2.** According to Braaksma (5, p. 278)

$$H^{m,\,n}_{p,\,q}\left[\, x \,\middle|\, \begin{matrix}\{(a_p,\, \alpha_p)\}\\ \{(b_q,\, \beta_q)\}\end{matrix}\right] = 0\ (\,|\,x\,|^{\,a}\,)\ \text{for small } x,$$

where $\quad \sum_1^p \alpha_j - \sum_1^q \beta_j \leqslant 0$ and $a = \min.\ Re\left(\dfrac{b_h}{\beta_h}\right)\ (h = 1, \ldots, m)$

and

$$H^{m,\,n}_{p,\,q}\left[\, z \,\middle|\, \begin{matrix}\{(a_p,\, \alpha_p)\}\\ \{b_q,\, \beta_q)\}\end{matrix}\right] = 0\ (\,|\,x\,|\,^{\beta})\ \text{for large } x,$$

where $\quad \sum_1^p \alpha_j - \sum_1^q \beta_j < 0,\ \sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0,$

$$|\arg x\,| < \tfrac{1}{2}\,\lambda\,\pi$$

and $\quad \beta \max Re\left(\dfrac{a_i - 1}{\alpha_i}\right)\ (i = 1, \ldots, n).$

Very recently Gupta and Jain (9) have shown that :

(2·1) If one of $(a_j, \alpha_j)\ (j = 1, \ldots, m)$ is the same as one of $(b_h, \beta_h)$ $(h = m + 1, \ldots, q)$ or one of $(b_h, \beta_h)\ (h = 1, \ldots, m)$ is the same as one of $(a_j, \alpha_j)\ (j = n + 1, \ldots, p)$, then the H-function reduces to one of the lower order *i. e.* each of $p$, $q$ and $n$ or $m$ decreases by unity.

(2·2) $$H^{m,\,n}_{p,\,q}\left[\, x \,\middle|\, \begin{matrix}\{(a_p,\, \alpha_p)\}\\ \{(b_q,\, \beta_q)\}\end{matrix}\right] \equiv H^{n,\,m}_{q,\,p}\left[\, \frac{1}{x} \,\middle|\, \begin{matrix}\{(1 - b_q,\, \beta_q)\}\\ \{(1 - a_p,\, \alpha_p)\}\end{matrix}\right].$$

(2·3) $$H^{m,\,n}_{p,\,q}\left[\, x \,\middle|\, \begin{matrix}\{(a_p,\, 1)\}\\ \{(b_q,\, 1)\}\end{matrix}\right] \equiv G^{m,\,n}_{p,\,q}\left[\, x \,\middle|\, \begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right].$$

**3.** In this section we shall evaluate an infinite integral by making use of a known result, due to Bhise (2, p. 72), given as below :

If

$$h(t) \doteqdot f(\sqrt{s})$$

$$\phi(s : \nu\,;\, \sigma) \underset{\nu}{\overset{K}{=}} t^{\sigma} . f(t)$$

and $$\phi(s : k,\, m\,;\, \sigma) \underset{m}{\overset{k}{=}} t^{\sigma} .\ h(1/t)$$

then [1]

$$\phi\, 2\sqrt{2s} : 2m ; -2k - 5/2) = \Gamma_*(\tfrac{1}{2} - k \pm m)\sqrt{\frac{2}{\pi s}}\, \phi(s : k, m ; -k - 7/4),$$

provided that $Re\,(k \pm m) < \frac{1}{2}$, the Laplace transform of $|\,h(t)\,|$ and the Whittaker transform of $|\,t^{-k-7/4}\, h(t)\,|$ exist.

Since

$$G_{0,\,1}^{1,\,0}\,(st \,|\, 0) \equiv e^{-s}, \quad Re\,(s) > 0,$$

the Laplace transform of

$$t^{-l} H_{p,\,q}^{\mu,\,\nu}\left[zt^{\sigma} \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right]$$

can be expressed by the integral

$$(3{\cdot}1) \qquad s\int_0^{\infty} t^{-l}\, G_{0,\,1}^{1,\,0}\left(st \,|\, 0\right) H_{p,\,q}^{\mu,\,\nu}\left[z\, t^{\sigma} \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] dt.$$

Using a result, due to Gupta and Jain(9);

$$\int_0^{\infty} t^{\eta-1}\, G_{r,\,l}^{k,\,f}\left(st \,\middle|\, \begin{matrix}c_1, \ldots, c_r\\ d_1, \ldots, d_l\end{matrix}\right) H_{p,\,q}^{m,\,n}\left[zt^{\sigma} \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] dt$$

$$= s^{-\eta}\, H_{p+l,\, q+r}^{m+f,\, k+n}\left[\frac{z}{s^{\sigma}} \,\middle|\, \begin{matrix}\{(a_n, \alpha_n)\}, \{(1-d_l-\eta, \sigma)\}, (a_{n+1}, \alpha_{n+1}), \ldots, (a_p, \alpha_p)\\ \{(b_m, \beta_m)\}, \{(1-c_r-\eta, \sigma)\}, (b_{m+1}, \beta_{m+1}), \ldots, (b_q, \beta_q)\end{matrix}\right],$$

provided that $Re\,[\eta + \sigma\left(\frac{b_h}{\beta_h}\right) + d_i] > 0 \; (h = 1, \ldots, m\,;\, i = 1, \ldots, k)$,

$Re\,[\eta + (c_j - 1) + \sigma\left(\frac{a_{h'} - 1}{\alpha_{h'}}\right)] < 0 \; (j = 1, \ldots, f\,;\, h' = 1, \ldots, n), \sigma > 0,$

$\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0,\; 2k + 2f - l - r \equiv \mu > 0,\; |\arg z| < \frac{1}{2}\lambda\pi$ and $|\arg s| < \frac{1}{2}\mu\pi$, we find that the integral (3·1) yields

$$f(\sqrt{s}) \equiv s^{l}\, H_{p+1,\,q}^{\mu,\,\nu+1}\left[z\, s^{-\sigma} \,\middle|\, \begin{matrix}(l, \sigma), \{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right]$$

$$\doteqdot t^{-l}\, H_{p,\,q}^{\mu,\,\nu}\left[z\, t^{\sigma} \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] \equiv h(t),$$

provided that $Re\,[1 - l + \sigma\left(\frac{b_h}{\beta_h}\right)] > 0 \; (h = 1, \ldots, \mu), \sigma > 0, Re\,(t) > 0$

---

(1) For the sake of brevity the symbol $\Gamma_*(a \pm b)$ has been used to denote $\Gamma(a+b)\;\Gamma(a-b)$.

$\sum_1^p a_j - \sum_1^q \beta_j \leqslant 0,\ \sum_1^\nu \alpha_j - \sum_{\nu+1}^p a_j + \sum_1^\mu \beta_j - \sum_{\mu+1}^q \beta_j \equiv \lambda > 0, |\arg z| < \frac{1}{2}\lambda\pi$ and $|\arg s| < \frac{1}{2}\pi$.

Now

$$(3{\cdot}2) \qquad \phi(s : 2m ; -2k - 5/2)$$

$$= \sqrt{\frac{2}{\pi}}\, S \int_0^\infty x^{2l-2k-5/2} \sqrt{sx}\, K_{2m}(sx)\, H^{\mu,\,\nu+1}_{p+1,\,q}\left[\frac{z}{x^{2\sigma}}\left|\begin{matrix}(l,\sigma), \{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right.\right] dx.$$

Integrating (3·2) with the help of the result due to Gupta (8, p. 99) :[1]

$$\int_0^\infty x^{\eta-1} K_\nu(sx)\, H^{m,n}_{p,q}\left[zx^\sigma\left|\begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right.\right] dx$$

$$= 2^{\eta-2} s^{-\eta} H^{m,\,n+2}_{p+2,\,q}\left[z(2/s)^\sigma\left|\begin{matrix}(1-\frac{1}{2}\eta \pm \frac{1}{2}\nu, \sigma), \{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right.\right],$$

where $Re\,(\eta + \sigma.b_h/\beta_h \pm \nu) > 0\ (h = 1, \ldots, m)$, $\sigma > 0$ $Re\,(s) > 0$ and

$$|\arg z| < \tfrac{1}{2}\lambda\pi,\ \sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0;$$

and replacing $s$ by $2\sqrt{2s}$ we get :

$$\phi(2\sqrt{2s} : 2m ; -2k - 5/2)$$

$$= \frac{1}{\sqrt{\pi}} (2s)^{5/4+k-l}\ H^{\mu+2,\,\nu+1}_{p+1,\,q+2}\left[z(2s)^\sigma\left|\begin{matrix}(l,\sigma), \{(a_p, \alpha_p)\}\\ (l-k-\frac{1}{2} \pm m, \sigma), \{(b_q, \beta_q)\}\end{matrix}\right.\right],$$

where $Re\,(k \pm m) < \frac{1}{2}$, $Re\left[l - k \pm m + \sigma\left(\frac{1-a_j}{\alpha_j}\right)\right] > \frac{1}{2}\ (j = 1, \ldots, \nu)$,

$Re\,(s) > 0$, $\sigma > 0$, $\sigma + \sum_1^\nu \alpha_j - \sum_{\nu+1}^p \alpha_j + \sum_1^\mu \beta_j - \sum_{\mu+1}^q \beta_j \equiv \lambda > 0$ and $|\arg z| < \frac{1}{2}\lambda\pi$

Then according to the result mentioned in the beginning of this section we obtain

$$(3{\cdot}3) \qquad \int_0^\infty (2st)^{-\frac{1}{4}} W_{k,m}(2st)\, t^{l-k-7/4}\, H^{\mu,\nu}_{p,q}\left[zt^{-\sigma}\left|\begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right.\right] dt$$

$$= \frac{1}{\Gamma_*(\frac{1}{2} - k \pm m)} (2s)^{3/4-l+k}$$

$$\times H^{\mu+2,\,\nu+1}_{p+1,\,q+2}\left[z(2s)^\sigma\left|\begin{matrix}(l,\sigma), \{(a_p, \alpha_p)\}\\ (l-k-\frac{1}{2} \pm m, \sigma), \{(b_q\ \beta_q)\}\end{matrix}\right.\right],$$

---

(1) For the sake of brevity the symbol $(a \pm b, \sigma)$ is used to denote a set of the parameters $(a+b,\sigma)$, $(a-b,\sigma)$.

provided that $Re\,(k \pm m) < \frac{1}{2}$, $Re\left[l - k \pm m + \sigma \left(\frac{1 - a_j}{\alpha_j}\right)\right] > \frac{1}{2}$ $(j = 1, \ldots, \nu)$

$Re\,(1 - l + \sigma .\, b_h/\beta_h) > 0$ $(h = 1, \ldots, \mu)$, $Re\,(s) > 0$, $\sigma > 0$, $\sum_1^p \alpha_j - \sum_1^q \beta_j \leqslant 0$,

$\sum_1^\nu \alpha_j - \sum_{\nu+1}^p \alpha_j + \sum_1^\mu \beta_j - \sum_{\mu+1}^q \beta_j \equiv \lambda > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$ and $|\arg s| < \frac{1}{2}\pi$.

**4. Theorem**

If $t^{(1-d)/2d} f(t)$, where $d \equiv 1/\gamma \left(\sum_1^q \beta_j - \sum_1^p \alpha_j\right) > 0$,

and $H[f(t)\,;\,\xi\,;\,(a_p, \alpha_p)\,;\,(b_q, \beta_q)]$ belong to $L\,(0, \infty)$, $Re\,(\mu) > -1$,

$Re\,(s) \geqslant s_0 > 0$, $Re\,(\lambda \pm \delta) < \frac{1}{2} Re\left(\frac{7\gamma - 2}{4\gamma} + \mu \pm \delta + \frac{b_h}{\beta_h}\right) > 0$ $(h = 1, \ldots, q)$,

$Re\left(\frac{5\gamma - 2}{4\gamma} + \lambda + \mu + \frac{a_i - 1}{\alpha_i}\right) < 0$ $(i = 1, \ldots, p)$ and let

$$\phi_\mu\,(s : \lambda, \delta) \frac{\lambda}{\overline{\overline{\delta}}}\; t^\mu f(t),$$

that is

$$\phi_\mu\,(s : \lambda, \delta) = \int_0^\infty (2st)^{-\frac{1}{2}}\, W_{\lambda,\delta}\,(2st)\, t^\mu f(t)\, dt$$

then

$$\phi_\mu\,(s : \lambda, \delta) = \int_0^\infty k\,(s, \xi)\, H\,[f(t)\,;\,\xi\,;\,(a_p, \alpha_p)\,;\,(b_q, \beta_q)]\, d\xi,$$

where

$$k\,(s, \xi) = \xi^{\frac{1-2\gamma}{\gamma} - \mu}\, \frac{1}{\Gamma_*(\frac{1}{2} - \lambda \pm \delta)} \times$$

$$\times H^{p+2,\, q+1}_{2q+2p+2}\left[\frac{2s}{\xi}\,\middle|\, \begin{matrix} (\frac{3}{4} + \lambda, 1), \{(1/4\gamma - \mu - b_q - \frac{2\gamma - 1}{4\gamma}\beta_q, \beta_q)\}, \\ (\frac{1}{4} \pm \delta, 1), \{(1/4\gamma - \mu - a_p - \frac{2\gamma - 1}{4\gamma}\alpha_p, \alpha_p)\}, \end{matrix}\right.$$

$$\left. \times \begin{matrix} \{(\frac{1 - 4\gamma}{4\gamma} - \mu + b_q + \frac{2\gamma + 1}{4\gamma}\beta_q, \beta_q)\} \\ \{(\frac{1 - 4\gamma}{4\gamma} - \mu + a_p + \frac{2\gamma + 1}{4\gamma}\alpha_p, \alpha_p)\} \end{matrix}\right]$$

**Proof**

Since $t^{(1-d)/2d} f(t)$ and $H[f(t)\,;\,\xi\,;\,(a_p, \alpha_p)\,;\,(b_q, \beta_q)]$ belong to $L\,(0, \infty)$ and the kernel in (1·5) is a Fourier kernel, therefore using a result due to Fox (7, p. 412), we have

$$f(t) = \int_0^\infty H\,[f(t)\,;\,\xi\,;\,(a_p,\,\alpha_p)\,;\,(b_q,\,\beta_q)]\;\xi^{\frac{3}{4\gamma}-1-}\;t^{\frac{1}{4\gamma}}$$

$$\times\; H\,{}^{q,\,p}_{2p,\,2q}\left[\;\xi t\left|\begin{matrix}\{(a_p+\frac{2\gamma-1}{4\,\gamma}\,\alpha_p,\,\alpha_p)\}\;,\;\{(1-a_p-\frac{2\gamma+1}{4\gamma}\,\alpha_p,\,\alpha_p)\}\\ \{(b_q+\frac{2\gamma-1}{4\gamma}\,\beta_q,\,\beta_q)\}\;,\;\{(1-b_q-\frac{2\gamma+1}{4\gamma}\,\beta_q,\,\beta_q)\}\end{matrix}\right.\right]\,d\xi,$$

provided that $d > 0$, $Re\,(b_h + \frac{2\gamma-1}{4\gamma}\,\beta_h) \geqslant \frac{\beta_h}{2\gamma}\,.\,\,(1-d)/2^d\;(h = 1, \ldots, q)$,

$Re\,(1-a_i-\frac{2\gamma-1}{4\gamma}\,\alpha_i) > \frac{\alpha_i}{2\gamma}\,.\,(1+d)/2^d\;(i = 1, \ldots, p)$ and $f(t)$ is of bounded variation near $t = \xi$ $(\xi > 0)$.

The Whittaker transform of $t^\mu f(t)$ is

$$\phi_\mu(s:\lambda,\delta) = \int_0^\infty (2st)^{-\frac{1}{2}}\;W_{\lambda,\delta}\;(2st)\;t^\mu\;dt\;\int_0^\infty H\,[f(t)\,;\,\xi\,;\,(a_p,\,\alpha_p)\,;\,(b_q,\,\beta_q)]\;\times$$

$$\times\; t^{-\frac{1}{4\gamma}}\;\xi^{\frac{3}{4\gamma}-1}\;H\,{}^{q,\,p}_{2p,\,2q}\left[\;\xi t\left|\begin{matrix}\{(a_p+\frac{2\gamma-1}{4\gamma}\,\alpha_p,\,\alpha_p)\},\;\{(1-a_p-\frac{2\gamma+1}{4\gamma}\,\alpha_p,\,\alpha_p)\}\\ \{(b_q+\frac{2\gamma-1}{4\gamma}\,\beta_q,\,\beta_q)\},\;\{(1-b_q-\frac{2\gamma+1}{4\gamma}\,\beta_q,\,\beta_q)\}\end{matrix}\right.\right]d\xi.$$

On changing the order of integration in the above integral we have :

$$(4{\cdot}1)\qquad \phi_\mu(s:\lambda,\delta) = \int_0^\infty \xi^{\frac{3}{4\gamma}-1}\;H\,[f(t)\,;\,\xi\,;\,(a_p,\,\alpha_p)\,;\,(b_q,\,\beta_q)]\;d\xi\;\times$$

$$\times\;\int_0^\infty (2st)^{-\frac{1}{2}}\;W_{\lambda,\delta}\,(2st)\;t^{\mu-1/(4\gamma)}\;H\,{}^{q,\,p}_{2p,\,2q}\left[\;\xi t\left|\begin{matrix}\{(a_p+\frac{2\gamma-1}{4\gamma}\,\alpha_p,\,\alpha_p)\},\\ \{(b_q,+\frac{2\gamma-1}{4\gamma}\,\beta_q,\,\beta_q)\}\end{matrix}\right.\right.$$

$$\left.\times\;\begin{matrix}\{(1-a_p-\frac{2\gamma+1}{4\gamma}\,\alpha_p,\,\alpha_p)\}\\ \{(1-b_q-\frac{2\gamma+1}{4\gamma}\,\beta_q,\,\beta_q)\}\end{matrix}\right]dt.$$

The theorem then follows directly on evaluating the $t$–integral with the help of (3·3) after replacing $\xi\,t$ by $t$ and making use of (2·2).

To justify the change of the order of integration in (4·1) we see that the $\xi$–integral is absolutely and uniformly convergent if the generalised Hankel transform of $|\,g(\xi)\,|$ exists ; and the $t$–integral is so if $Re\,(\lambda \pm \delta) < \frac{1}{2}$,

$$Re\left(\frac{7\gamma-2}{4\gamma}+\mu\pm\delta+\frac{b_h}{\beta_h}\right)>0\ (h=1,\ldots,q),\ Re\left(\frac{5\gamma-2}{4\gamma}+\lambda+\mu+\frac{a_i-1}{\alpha_i}\right)<0$$

$(i=1,\ldots,p)$ and $|\arg\xi|>0$. Hence by Fubini's theorem the inversion of the order of—integration is justified.

4·1. *Particular cases :*

The transformation defined in (1·5) can be reduced to various generalisations, given earlier by Watson (19, p. 308), Bhatnagar (1, p. 43), R. Narain {(11, p. 271) and (12, p. 298)} and Everitt (6, p. 271). Therefore our theorem will yield many new results, as particular cases, by making suitable substitutions. However, some of the known results, obtained as the particular cases, are mentioned here.

Taking $\gamma=1, p=1, q=2, \alpha_1=\beta_1=\beta_2=1, a_1=k-m-\frac{1}{2}-\nu/2, b_1=\nu/2$ and $b_2=\nu/2+2m$, the above theorem yields a result, due to Saxena (15, p. 302).

With $\gamma=1, \lambda=\frac{1}{4}, \delta=\pm\frac{1}{4}, p=0, q=1, \beta_1=1, b_1=\nu/2$ the above theorem reduce to a result given by Bhonsle (3, p. 114).

**5.** *Certain infinite integrals.* In our further discussion, because of large number of parameters, the notation

$\left(\Delta\left(\delta, a-\begin{vmatrix} r_1\\ r_2\\ \vdots\\ r_n\end{vmatrix}\right), 1\right)$ will stand to represent a set of the parameters

$(\Delta(\delta, a-r_1), 1), \ldots, (\Delta(\delta, a-r_n), 1)$.

By using the theorem, established above in the section 4, we can obtain many new infinite integrals. However, a few interesting cases, which are believed to be new and give generalisations of certain known integrals, are mentioned as under :

$$(5{\cdot}1)\qquad \int_0^\infty (2st)^{-\frac{1}{4}}\,W_{\lambda,\delta}(2st)\,e^{-\sigma t}\,t^{\mu+\rho-1}\,E(\lambda_1,\lambda_2::\sigma t)\,E(\mu_1,\mu_2::\nu t)\,dt$$

$$=(2s)^{-\mu}\sigma^{-\rho}\Gamma(\lambda_2)\cdot\sum_{r=0}^{\infty}\frac{\Gamma(\lambda_1+r)}{r!}\sum_{u=0}^{\infty}\frac{\Gamma(\mu_1+u)\Gamma(\mu_2+r+u)}{u!}\left(\frac{\nu-\sigma}{\nu}\right)^u$$

$$\times\frac{1}{\Gamma_*(\frac{1}{4}-\lambda\pm\delta)}\,G^{2,4}_{4,4}\left[\frac{2s}{\sigma}\left|\begin{matrix}\left(1-\rho-\begin{vmatrix}\lambda_1+\mu_1\\ \lambda_2+\mu_2\\ \lambda_1+\mu_2+r\end{vmatrix}\right),(\frac{3}{4}+\lambda+\mu)\\ (\frac{1}{4}+\mu\pm\delta),\left(1-\rho-r-\lambda_1-\mu_2-\begin{vmatrix}\lambda_2\\ \mu_1+u\end{vmatrix}\right)\end{matrix}\right.\right]$$

provided that $Re(\sigma)>0$, $Re(\nu)>\frac{1}{2}Re(\sigma)$, $Re(\mu)>-1$, $Re(s)>0$, $Re(\lambda\pm\delta)<\frac{1}{2}$ and $Re(\rho+\mu\pm\delta+\lambda_i+\mu_j)>-\frac{1}{4}\ (i,j=1,2)$.

(5·2) $$\int_0^\infty (2st)^{-\frac{1}{4}} W_{\lambda,\delta}(2st)\, e^{-\frac{1}{2}t} W_{k,l}(t) \sin(c\sqrt{t})\, t^{\mu+\rho-1}\, dt$$

$$= \frac{c\,(2s)^{-\mu}}{\Gamma_*(\frac{1}{2}-\lambda\pm\delta)} \sum_{r=0}^{\infty} \frac{(-c^2/4)^r}{r!\,(3/2)_r}\, G^{2,3}_{3,3}\left[2s \,\middle|\, \begin{matrix}(-\rho-r\pm l), (\frac{3}{4}+\lambda+\mu) \\ (\frac{1}{4}+\mu\pm\delta), (k-\rho-r-\frac{1}{2})\end{matrix}\right],$$

provided that $Re(\mu) > -1$, $Re(s) > 0$, $Re(\lambda\pm\delta) < \frac{1}{2}$, and

$$Re(\rho+\mu\pm l\pm\delta) > -5/4.$$

(5·3) $$\int_0^\infty (2st)^{-\frac{1}{4}} W_{\lambda,\delta}(2st)\, e^{-\frac{1}{2}t} W_{k,l}(t) \cos(c\sqrt{t})\, t^{\mu+\rho-1}\, dt$$

$$= \frac{(2s)^{-\mu}}{\Gamma_*(\frac{1}{2}-\lambda\pm\delta)} \sum_{r=0}^{\infty} \frac{(-c^2/4)^r}{r!\,(\frac{1}{2})_r}\, G^{2,3}_{3,3}\left[2s \,\middle|\, \begin{matrix}(\frac{1}{2}-\rho-r\pm l), (\frac{3}{4}+\lambda+\mu) \\ (\frac{1}{4}+\mu\pm\delta), (k-\rho-r)\end{matrix}\right],$$

provided that $Re(\mu) > -1$, $Re(s) > 0$, $Re(\lambda\pm\delta) < \frac{1}{2}$ and

$$Re(\rho+\mu\pm l\pm\delta) > -\tfrac{3}{4}.$$

In the proof we shall require the following result due to Gupta and Jain(9)

(5·4) $$\int_0^\infty x^{\eta-1} H^{m,n}_{p,q}\left[z x^\sigma \,\middle|\, \begin{matrix}\{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\}\end{matrix}\right] H^{k,f}_{r,l}\left[sx \,\middle|\, \begin{matrix}\{(c_r,\gamma_r)\} \\ \{(d_l,\delta_l)\}\end{matrix}\right] dx$$

$$= s^{-\eta} H^{m+f,\,n+k}_{p+l,\,q+r}\left[\frac{z}{s^\sigma} \,\middle|\, \begin{matrix}\{(a_n,\alpha_n)\}, \{(1-d_l-\eta\delta_l, \sigma\delta_l)\}, \\ \{(b_m,\beta_m)\}, \{(1-c_r-\eta\gamma_r, \sigma\gamma_r)\},\end{matrix}\right.$$

$$\left.\times \begin{matrix}(a_{n+1},\alpha_{n+1}),\ldots,(a_p,\alpha_p) \\ (b_{m+1},\beta_{m+1}),\ldots,(b_q,\beta_q)\end{matrix}\right],$$

provided that $Re\left(\eta+\sigma\dfrac{b_h}{\beta_h}+\dfrac{d_i}{\delta_i}\right) > 0$ $(h=1,\ldots,m;\ i=1,\ldots,k)$,

$Re\left(\eta+\dfrac{c_j-1}{\gamma_j}+\sigma\dfrac{a_{h'}-1}{\alpha_{h'}}\right) < 0$ $(j=1,\ldots,f;\ h'=1,\ldots,n)$, $\sigma>0$.

$|\arg z| < \frac{1}{2}\lambda\pi$, $|\arg s| < \frac{1}{2}\mu\pi$, where $\sum_1^n \alpha_j - \sum_{n+1}^{p}\alpha_j + \sum_1^m \beta_j - \sum_{m+1}^{q}\beta_j \equiv \lambda > 0$

and $\sum_1^f \gamma_j - \sum_{f+1}^{r}\gamma_j + \sum_1^k \delta_j - \sum_{k+1}^{l}\delta_j \equiv \mu > 0.$

Using a known result(16).

$$\int_0^\infty x^{\rho-1} e^{-\sigma x} E(\lambda_1,\lambda_2::\sigma x)\, E(\mu_1,\mu_2::vx)\, H^{m,n}_{p,q}\left[z x^{\delta/t} \,\middle|\, \begin{matrix}\{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\}\end{matrix}\right] dx$$

$$= (2\pi)^{\frac{1}{2}(1-\delta)+(1-t)\ (m+n-\frac{1}{2}p-\frac{1}{2}q)}\ t^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}p - \frac{1}{2}q + 1}\ \delta^{\rho-\frac{1}{2}}\ \sigma^{-\rho}\,\Gamma(\lambda_2) \times$$

$$\times \sum_{r=0}^{\infty} \frac{\Gamma(\lambda_1+r)\,\delta^{-r}}{r!} \sum_{u=0}^{\infty} \frac{\Gamma(\mu_1+u)\,\Gamma(\mu_2+r+u)}{u!} \left(\frac{\nu-\sigma}{\nu\delta}\right)^u$$

$$\times H^{tm,\ tn+3\delta}_{tp+3\delta,\ tq+2\delta}\left[(z.\ t^{\tau})^t \left(\frac{\delta}{\sigma}\right)^{\delta} \left| \begin{matrix} \left(\Delta\left(\delta,\ 1-\rho-\left|\begin{matrix}\lambda_1+\mu_1\\ \lambda_2+\mu_2\\ \lambda_1+\mu_2+r\end{matrix}\right|\right),\ 1\right), \\ \{(\Delta(t,\ b_q),\ \beta_q)\}, \end{matrix}\right.\right.$$

$$\left.\left. \begin{matrix} \{(\Delta(t,\ a_p),\ a_p)\} \\ \times\left(\Delta\left(\delta,\ 1-\rho-r-\lambda_1-\mu_2-\left|\begin{matrix}\lambda_2\\ \mu_1+u\end{matrix}\right|\right),\ 1\right)\end{matrix}\right.\right],$$

where $\delta$ and $t$ are positive integers, $Re\ (\sigma) > 0$, $Re\ (\nu) > \frac{1}{2}\ Re\ (\sigma)$,

$\sum_1^p a_j - \sum_1^q \beta_j \equiv \tau \leqslant 0$, $\sum_1^n a_j - \sum_{n+1}^p a_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$ and

$Re\ (\rho + \lambda_i + \mu_j + \frac{\delta}{t}\frac{b_h}{\beta_h}) > 0\ (i = 1, 2\ ;\ j = 1, 2\ ;\ h = 1, \ldots, m)$,

we have

$$H\,[e^{-\sigma t}\, t^{\rho-1} E\,(\lambda_1, \lambda_2 :: \sigma t)\, E\,(\mu_1, \mu_2 :: \nu t)\ ;\ \xi\ ;\ (a_p, a_p)\ ;\ (bq, \beta q)]$$

$$= \xi^{-1/(4\gamma)}\,\sigma^{-3/(4\gamma)-\rho+1}\,\Gamma(\lambda_2) \sum_{r=0}^{\infty} \frac{\Gamma(\lambda_1+r)}{r!} \sum_{u=0}^{\infty} \frac{\Gamma(\mu_1+u)\,\Gamma(\mu_2+r+u)}{u!} \left(\frac{\nu-\sigma}{\nu}\right)^u \times$$

$$\times H^{q,\ p+3}_{2p+3,\ 2q+2}\left[\frac{\xi}{\sigma} \left| \begin{matrix} \left(2-\frac{3}{4\gamma}-\rho-\left|\begin{matrix}\lambda_1+\mu_1\\ \lambda_2+\mu_2\\ \lambda_1+\mu_2+r\end{matrix}\right|, 1\right),\ \{(a_p+\frac{2\gamma-1}{4\gamma}a_p,\ a_p)\}, \\ \{(b_q+\frac{2\gamma-1}{4\gamma}\beta_q,\ \beta_q)\},\ \{(1-b_q-\frac{2\gamma+1}{4\gamma}\beta_q,\ \beta_q)\}, \end{matrix}\right.\right.$$

$$\left.\left. \begin{matrix} \{(1-a_p-\frac{2\gamma+1}{4\gamma}a_p,\ a_p)\} \\ \times\left(2-\frac{3}{4\gamma}-\rho-r-\lambda_1\,\mu_2-\left|\begin{matrix}\lambda_2\\ \mu_1+u\end{matrix}\right|,\ 1\right)\end{matrix}\right.\right],$$

where $Re\ (\sigma) > 0$, $Re\ (\nu) > \frac{1}{2}\ Re\ (\sigma)$, $\sum_1^p a_j - \sum_1^q \beta_j \leqslant 0$ and

$$Re\ (\rho + \lambda_i + \mu_j + \frac{b_h}{\beta_h}) > \frac{\gamma-1}{2\gamma}\ (i, j = 1, 2\ ;\ h = 1, \ldots, q).$$

Substituting this in the theorem, putting $1 + \mu = 1/(4\gamma)$, applying the identity (2·2), making use of (5·4) and using (2·1) the integral (5·1) can easily be obtained.

The integrals (5·2) and (5·3) can easily be established, if we proceed on similar lines and use the following known results(17) ;

$$\int_0^\infty x^{\rho-1} e^{-\frac{1}{2}x} \sin (c\sqrt{x})\, W_{k,\mu}(x)\, H_{p,q}^{m,n}\left[ z x^{\delta/t} \,\middle|\, \begin{matrix} \{(a_p, \alpha_p)\} \\ \{b_q, \beta_q)\} \end{matrix} \right] dx$$

$$= (2\pi)^{\frac{1}{2}(1-\delta) + (1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)}\, t^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}p - \frac{1}{2}q + 1} \,.\, c\, \delta^{\rho+k}$$

$$\times \sum_{r=0}^\infty \frac{(-c^2/4)^r\, \delta^r}{r!\,(3/2)_r}$$

$$\times H_{tp+2\delta,\, tq+\delta}^{tm,\, tn+2\delta}\left[ (zt^\tau)^t \delta^\delta \,\middle|\, \begin{matrix} (\Delta(\delta, -\rho-r\pm\mu), 1), \{(\Delta(t, a_p), \alpha_p)\} \\ \{(\Delta(t, b_q), \beta_q)\}, (\Delta(\delta, -\rho-r+k-\frac{1}{2}), 1) \end{matrix} \right],$$

where $\delta$ and $t$ are positive integers, $\sum_1^p \alpha_j - \sum_1^q \beta_j \equiv \tau \leqslant 0$, $\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$ and $Re\left(\rho + \frac{\delta}{t}\cdot\frac{b_h}{\beta_h}\right) > |Re\,\mu| - 1$ $(h = 1, \ldots, m)$,

and

$$\int_0^\infty x^{\rho-1} e^{-\frac{1}{2}x} \cos (c\sqrt{x})\, W_{k,\mu}(x)\, H_{p,q}^{m,n}\left[ z x^{\delta/t} \,\middle|\, \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right] dx$$

$$= (2\pi)^{\frac{1}{2}(1-\delta) + (1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)}\, t^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}p - \frac{1}{2}q + 1} \,.\, \delta^{\rho+k-\frac{1}{2}}$$

$$\times \sum_{r=0}^\infty \frac{(-c^2/4)^r\, \delta^r}{r!\,(\frac{1}{2})_r} \times$$

$$\times H_{tp+2\delta,\, tq+\delta}^{tm,\, tn+2\delta}\left[ (zt^\tau)^t \delta^\delta \,\middle|\, \begin{matrix} (\Delta(\delta, \frac{1}{2}-\rho-r\pm\mu), 1), \{(\Delta(t, a_p), \alpha_p)\} \\ \{(\Delta(t, b_q), \beta_q)\}, (\Delta(\delta, k-\rho-r), 1) \end{matrix} \right],$$

where $\delta$ and $t$ are positive integers, $\sum_1^p \alpha_j - \sum_1^q \beta_j \equiv \tau \leqslant 0$, $\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{n+1}^q \beta_j \equiv \lambda > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$, $R_e\left(\rho + \frac{\delta}{t}\frac{b_h}{\beta_h}\right) > |Re\,\mu| - \frac{1}{2} (h = 1, \ldots, m)$.

## Acknowledgements

I wish to express my sincere thanks to Dr. R. K. Saxena of G. S. Technological Institute, Indore for his kind help and guidance during the preparation of this paper. I am also thankful to the referee for some useful suggestions for revising the paper.

## References

1. Bhatnagar. K. P. Certain theorems on self-reciprocal functions. *Bull. Cal, Math. Soc.* **47** : 43-52, (1955).

2. Bhise. V. M. Some relations involving Meijer's K-transform. *J. Vikram Univ.* **9** : 68-77 (1960).

3. Bhonsle, B. R. A relation between Laplace and Hankel-transform. *Proc. Glasgow Math. Assoc.*, **5** : 114-115, (1962).

4. Bose, S. K. A study of generalised Laplace Integral I. *Bull. Cal. Math. Soc*, **41** : 9-27, (1949).

5. Braaksma, B. L. J. Asymptotic expansions and analytic-continuations for a class of Barnes integrals. *Compos. Math.*, **15** : 239-341, (1963).

6. Everitt. W. N. On a generalisation of Bessel functions and a resulting class of Fourier kernels. *Quart. J. Math.*, *Oxford*, **10, II** : 270-279, (1959).

7. Fox, C. The G and H-functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc*, **98** : 395-429, (1961).

8. Gupta, K. C. On the H-function. *Annales de la Societe' Scientifique de Bruxelles*, **79** : 98-104, (1965).

9. Gupta, K. C., and Jain, U. C. The H-function II. *To appear in Proc. Nat. Acad. Sci. India*

10. Meijer, C. S. Uber eine Erweiterung der Laplace transformatior. *Proc. Nedrl. Akad. Wet. Amsterdam*, **43** · 599-608, (1940).

11. Narain, R. On a generalisation of Hankel transform and self-reciprocal functions. *Rendi. Sem. Mat. Torino*, **16** : 269-300, (1956-57).

12. Narain, R. A Fourier kernel. *Math. Zeitschr.*, **70** : 297-299, (1959).

13. Narain, R. The G-functions as unsymmetrical Fourier kernels I. *Proc. Amer. Math. Soc*, **13** : 950-959, (1962).

14. Rathie. C. B. A theorem in operational calculus and some integrals involving Legendre, Bessel and E-functions. *Proc. Glasgow Math. Assoc.*, **2 IV** : 173-179, (1956).

15. Saxena, R. K. Relation between Whittaker transform and modified $x_{\nu, k, m}$ — transform. *Math. Annalen*, **154** : 301-306, (1964).

16. Sharma, O. P. Certain infinite integrals involving H-function and MacRobert's E-functions. *Communicated for publication in Proc. Nat. Acad. Sci., India.*

17. Sharma, O. P. Certain infinite and finite integrals involving H-function and confluent hypergeometric functions. *Accepted for publication in Proc. Nat Acad. Sci., India.*

18. Verma, R. S. A generalisation of Laplace transform. *Current Sci., India,* **16** : 17-18, (1947).

19. Watson, G. N. Some self-reciprocal functions. *Quart. J. Math., Oxford*, **2** (I) : 298-309, (1931].

# Application of Jacobi Polynomials to Nonlinear Oscillations—I—Free Oscillations

*By*

R. M. GARDE

*Department of Applied Mathematics, Government Engineering College, Jabalpur, M. P.*

[Received on 24th September, 1966]

**Abstract**

In this paper the author has linearised the nonlinear ordinary differential equation governing free oscillations by the linear Jacobi Polynomical approximation of the nonlinear restoring force or torque. The results thus obtained have been compared with exact results, where-ever possible.

## 1. Introduction

Recently Denman and Howard[1] and Denman and Liu[2] have used ultraspherical polynomials to solve some nonlinear free oscillation problems. The author[3,4] has applied Gegenbauer polynominals to some forced oscillation problems. This was achieved by obtaining an amplitude dependent approximation of the restoring force or torque and linearising the nonlinear differential equation governing the oscillations.

In the present paper a more general class of polynomials,— the Jacobi polynomials, are used to obtain a linear amplitude dependent approximation for the purpose of linearisation of the nonlinear differential equation governing free oscillations.

**2.** *Jacobi polynomials.* Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are sets of polynomials orthogonal in the interval $(-1, 1)$ with respect to the weight factor $(1-x)^\alpha (1+x)^\beta$, each set corresponding to values of $\alpha$ and $\beta$ such that $Re\,\alpha > -1$, $Re\,\beta > -1$. They may be obtained from : [5, p. 271]

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = 2^{\alpha+\beta}\rho^{-1}(1+t+\rho)^{-\beta}(1-t+\rho)^{-\alpha} \tag{1}$$

where

$$\rho = (1-2xt+t^2)^{\frac{1}{2}} \tag{2}$$

The ultraspherical, Gegenbauer, Legendre and Chebyshev polynomials are particular cases of Jacobi polynomials.

In the interval $(-A, A)$, Jacobi polynomials are defined as sets of polynomials orthogonal in this interval with respect to the weight factor $(1-x/A)^\alpha (+x/A)^\beta$. This gives rise to the polynomials $P_n^{(\alpha,\beta)}(x/A)$

**3.** *The General free Oscillation Problem.* A general free oscillation problem is characterised by the differential equation

$$\ddot{x} + f(x) = 0 \tag{3}$$

where dots represent differentiation with respect to time $t$. The function $f(x)$ is the nonlinear function related to the torque, force or voltage in a nonlinear physical problem. In the present paper $f(x)$ is assumed to be an odd function although the analysis is equally applicable to other cases. The general initial conditions for the oscillatory case may be taken as $x = A$ and $\dot{x} = 0$ at $t = 0$ or that $x = 0$ and $\dot{x} = v_0$ at $t = 0$, $A$ being the amplitude of the motion. A first integral of (3) is

$$\tfrac{1}{2}(\dot{x})^2 + V(x) = E \tag{4}$$

Where $V(x)$ is a potential function $[dv/dx = f(x)]$ and $E$ is proportional to the constant energy of the system. The periodic time of oscillation $T$ for the motion represented by (3) can be written as

$$\frac{T}{4} = \int_0^A [2(E - V)]^{-\frac{1}{2}}\, dx \tag{5}$$

where $T$ is in general a function of $A$. Since $E = V(A)$, we may write

$$\frac{T}{2^{3/2}} = \int_0^A [V(A) - V(x)]^{-\frac{1}{2}}\, dx \tag{6}$$

4. *Application of Jacobi Polynomial to free Oscillation Problem.*

For a function $f(x)$ which is expandable in terms of these polynomials in the interval $(-A, A)$ one obtains

$$f(x) = \sum_{n=0}^{\infty} a_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(x/A) \tag{7}$$

where the coefficients $a_n^{(\alpha,\beta)}$ are given by

$$a_n^{(\alpha,\beta)} = \frac{\int_{-1}^{1} f(Ax)\, P_n^{(\alpha,\beta)}(x)\,(1-x)^\alpha (1+x)^\beta\, dx}{\int_{-1}^{1} [P_n^{(\alpha,\beta)}(x)]^2 (1-x)^\alpha (1+x)^\beta\, dx} \tag{8}$$

If the series (7) is truncated after the second term, one obtains a linear approximation

$$f_*(x) = a_0^{(\alpha,\beta)} P_0^{(\alpha,\beta)}(x/A) + a_1^{(\alpha,\beta)} P_1^{(\alpha,\beta)}(x/A) \tag{9}$$

where star denotes approximation. The approximate period $T_*$ corresponding to the linear approximation is given by

$$T_* = \frac{4\pi}{(2+\alpha+\beta)} \left[\frac{Ag_1}{R}\right]^{\frac{1}{2}} \tag{10}$$

where

$$g_1 = \frac{2^{1+\alpha+\beta}\,\Gamma(2+\alpha)\,\Gamma(2+\beta)}{(3+\alpha+\beta)\,\Gamma(2+\alpha+\beta)} \tag{11}$$

and

$$R = \int_{-1}^{1} x f(Ax)\,(1-x)^\alpha (1+x)^\beta\, dx \tag{12}$$

## 5. *Application of linear Jacobi approximation to some typical problems.*

In the present section we shall apply Jacobi polynomial approximation to differential equations involving cubic, sine and hyperbolic sine nonlinearities.

(*i*) *Cubic nonlinearity* : Here we shall assume that $f(x) = ax + bx^3$, so that the differential equation for free oscillations is

$$\ddot{x} + ax + bx^3 = 0 \tag{13}$$

Equation (13) can be classified according to the signs of $a$ and $b$ as shown in table 1[2]. Only the first three cases yield bounded oscillatory motion. The exact solutions in all these three cases are known in terms of Jacobian elliptic functions.

TABLE 1

| Classification | Nomencleture | Sign of $a$ | Sign of $b$ | Motion |
|---|---|---|---|---|
| Case 1 | Hardening | + | + | Oscillatory |
| Case 2 | Softening | + | − | Conditionally oscillatory |
| Case 3 | Softening-Hardening | − | + | Oscillatory |
| Case 4 | Softening-Softening | − | − | Non-oscillatory |

We shall now discuss each case separately.

**Case 1.** *Hardening cubic* : The linear Jacobi polynomial approximation of $f(x)$ gives

$$f_*(x) = (ax + bx^3)_* = a_0^{(\alpha,\beta)} P_0^{(\alpha,\beta)}(x/A) + a_1^{(\alpha,\beta)} P_1^{(\alpha,\beta)}(x/A)$$
$$= a_0^{(\alpha,\beta)} + a_1^{(\alpha,\beta)} \left[ \frac{\alpha-\beta}{2} + \frac{2+\alpha+\beta}{2} \frac{x}{A} \right] \tag{14}$$

where $$a_0^{(\alpha,\beta)} = \frac{\int_{-1}^{1} (a\,Ax + bA^3 x^3)\, P_0^{(\alpha,\beta)}(x)\,(1-x)^\alpha (1+x)^\beta\,dx}{\int_{-1}^{1} [P_0^{(\alpha,\beta)}(x)]^2 (1-x)^\alpha (1+x)^\beta\,dx}$$

and $$a_1^{(\alpha,\beta)} = \frac{\int_{-1}^{1} (a\,Ax + bA^3 x^3)\, P_1^{(\alpha,\beta)}(x)\,(1-x)^\alpha (1+x)^\beta\,dx}{g_1} \tag{15}$$

Varma[6] has shown that

$$\int_{-1}^{1} x^s (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x)\,dx$$
$$= \frac{2^{n+\alpha+\beta+1}\, s!\, \Gamma(1+\alpha+n)\,\Gamma(1+\beta+s)}{n!\,(s-n)!\,\Gamma(n+s+\alpha+\beta+2)}\; {}_2F_1 \left[ \begin{matrix} -s+n,\ 1+\alpha+n\,; \\ -\beta-s\,; \end{matrix} \; -1 \right], \quad s > n \tag{16}$$

we also have the result [5, p. 261]

$$\int_{-1}^{1} x^s (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x)\,dx = 0, \quad s < n \tag{17}$$

and Bhonsle [7, p. 160] has shown that

$$\int_{-1}^{1} x^n (1-x)^\alpha (1+x)^\beta P^{n\,(\alpha,\beta)}(x)\,dx$$
$$= \frac{2^{1+\alpha+\beta+n}\,\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)}{\Gamma(2+\alpha+\beta+2n)} \tag{18}$$

Using (16), (17) and (18), we find that

$$a_0^{(\alpha,\beta)} = \frac{(\beta-\alpha)A}{(\alpha+\beta+2)}\left[a + \frac{(\alpha-\beta)^2+3(\alpha+\beta)+8}{(\alpha+\beta+3)(\alpha+\beta+4)}\,bA^2\right]$$

and
$$a_1^{(\alpha,\beta)} = \frac{1}{g_1}\left\{\frac{aA\;2^{2+\alpha+\beta}\,\Gamma(2+\alpha)\,\Gamma(2+\beta)}{\Gamma(4+\alpha+\beta)} + \frac{3bA^3\;2^{2+\alpha+\beta}\,\Gamma(2+\alpha)\,\Gamma(4+\beta)}{\Gamma(6+\alpha+\beta)}\right.$$
$$\left.\times\, {}_2F_1\left[\begin{matrix}-2,\,2+\alpha\,; \\ -\beta-3\,;\end{matrix}\; -1\right]\right\} \tag{19}$$

Replacing $f(x)$ by its approximation $f_*(x)$, the differential equation becomes

$$\ddot{x} + a_0^{(\alpha,\beta)} + a_1^{(\alpha,\beta)}\left\{\frac{\alpha-\beta}{2} + \frac{2+\alpha+\beta}{2}\,\frac{x}{A}\right\}$$

or
$$\ddot{x} + \omega^2_*\,x = -\frac{(\alpha-\beta)A}{(2+\alpha+\beta)}(\omega^2_* - \omega_1^2) \tag{20}$$

where
$$\omega_1^2 = a + \frac{bA^2\{(\alpha-\beta)^2+3(\alpha+\beta)+8\}}{(\alpha+\beta+3)(\alpha+\beta+4)}$$

and
$$\omega_*^2 = a_1^{(\alpha,\beta)}\left(\frac{2+\alpha+\beta)}{2A}\right)$$
$$= a + \frac{3bA^2\{(\alpha-\beta)^2+\alpha+\beta+4\}}{(5+\alpha+\beta)(4+\alpha+\beta)} \tag{21}$$

The approximate solution subject to the initial conditions $x = A$, $\dot{x} = 0$ at $t = 0$, is

$$x_* = \left[A + \frac{(\alpha-\beta)(1-\omega_1^2/\omega_*^2)}{(2+\alpha+\beta)}A\right]\cos\omega_* t - \frac{(\alpha-\beta)A}{2+\alpha+\beta}\left(1-\frac{\omega_1^2}{\omega_*^2}\right) \tag{22}$$

The approximate period of oscillation is given by

$$T_* = \frac{2\pi}{\omega_*} = \frac{2\pi}{\left[a + \frac{3bA^2\{(\alpha-\beta)^2+\alpha+\beta+4\}}{(\alpha+\beta+4(\alpha+\beta+5)}\right]^{\frac{1}{2}}} \tag{23}$$

The exact solution in given by [8, p. 40]

$$x = Ac_n\{(a+bA^2)^{\frac{1}{2}}\,t\} \tag{24}$$

where $cn\,(a+bA^2)^{\frac{1}{2}}\,t$ is the Jacobian elliptic function of modulus $k$ given by

$$k^2 = \frac{bA^2}{2(a+bA^2)} \tag{25}$$

The exact period is given by

$$T = \frac{4\,F\,(k, \frac{1}{2}\pi)}{(a + bA^2)^{\frac{1}{2}}} \tag{26}$$

where $F\,(k, \frac{1}{2}\pi)$ is the complete elliptic integral of the first kind of modulus $k$.

The limiting behaviour of both $T$ and $T_*$ is found to be similar as $A \to 0$ or $A \to \infty$

We find that as $A \to 0$

$$T \to \frac{6{\cdot}28}{a^{\frac{1}{2}}}$$

and
$$T_* \to \frac{2\pi}{a^{\frac{1}{2}}} = \frac{6{\cdot}28}{a^{\frac{1}{2}}} \text{ for finite } \alpha \text{ and } \beta.$$

Also as $A \to \infty$, both $T$ and $T_* \to 0$.

Denoting the nonlinearity factor $\frac{bA^2}{a}$ by $\nu$ we find that as $\nu \to 0$ both $T$ and $T_* \to \frac{2\pi}{a^{\frac{1}{2}}}$

Also as $\nu \to \infty$ both $T$ and $T_* \to 0$.

In order to compare the approximate results with the exact results, we consider a numerical case. For this case let $a = 10$, $b = 100$ and $A = 1$. The exact solution becomes

$$x = cn\,\sqrt{110}\,t \tag{27}$$

Also
$$k = \left(\frac{100}{2x\,110}\right)^{\frac{1}{2}} = 0{\cdot}674 \tag{28}$$

and
$$T = \frac{4F\,(0{\cdot}674, \frac{1}{2}\pi)}{(110)^{\frac{1}{2}}} = 0{\cdot}695 \tag{29}$$

For the approximate solution, let $\alpha = -\,0{\cdot}4$ and $\beta = -\,0{\cdot}5$ so that (11) gives

$$g_1 = 0{\cdot}4247 \tag{30}$$

and
$$\omega_*^2 = 83{\cdot}42 \text{ and } \omega_1^2 = 91{\cdot}57 \tag{31}$$

The approximate solution is thus

$$x_* = 0{\cdot}99 \cos 9{\cdot}133\,t + 0{\cdot}01 \tag{32}$$

and the approximate period is

$$T_* = \frac{2\pi}{\omega_*} = 0{\cdot}688 \tag{33}$$

A graphical comparison of the approximate and exact solutions is exhibited in fig. 1. The variation of the dimensionless period $T/T'$ with the nonlinearity factor $\nu$ is shown in fig. 2, $T$ being the period of the corresponding linear system *i. e.* $T' = \frac{2\pi}{a^{\frac{1}{2}}}$

EXACT SOLUTION OF THE DIFFERENTIAL EQUATION $\ddot{X} + 10X + 100X^3 = 0$, SUBJECT TO THE INITIAL CONDITIONS $X = 1$ AND $\dot{X} = 0$ ALONG WITH THE APPROXIMATE SOLUTION CORRESPONDING TO $\alpha = -0.4$ AND $\beta = -0.5$.

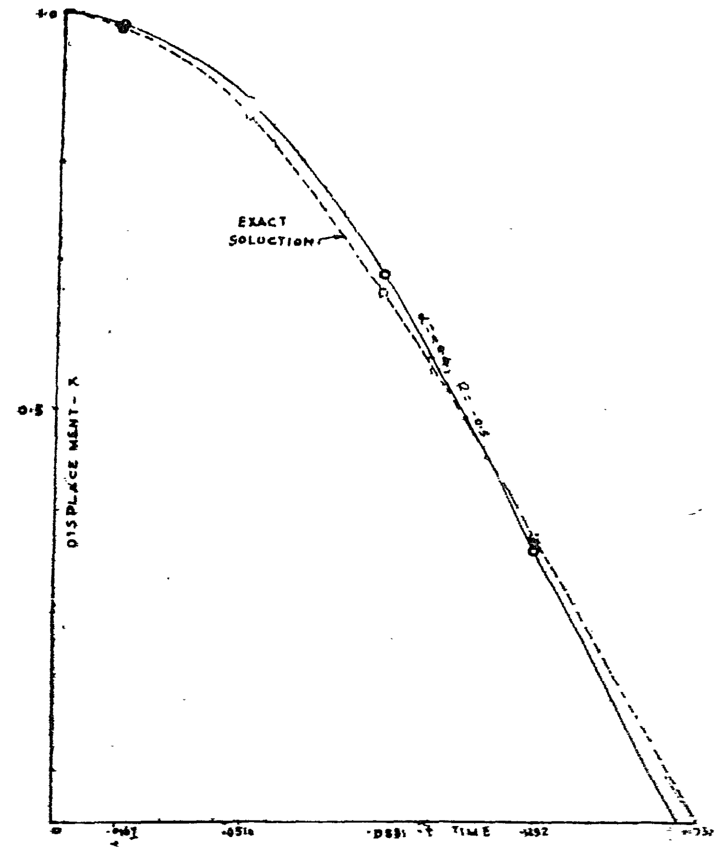



FIG 1

VARIATION OF THE DIMENSIONLESS PERIOD $T/T'$ WITH NONLINEARITY FACTOR $\nu$ FOR THE DIFFERENTIAL EQUATION
$\ddot{X} + 10X + 100X^3 = 0$

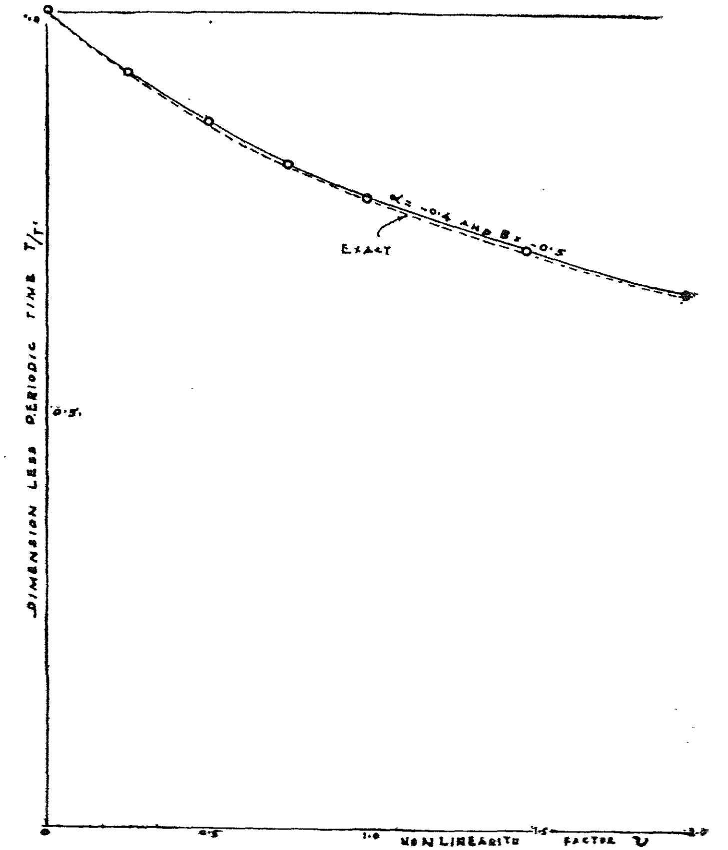



FIG 2.

**Case 2.** *Softening cubic* : In this case let $f(x) = ax - bx^3$ $(a > 0, b > 0)$ so that the differential equation is

$$\ddot{x} + ax - bx^3 = 0 \tag{34}$$

If the initial conditions are $x = 0$ and $\dot{x} = v_0$ at $t = 0$, then for a stable periodic motion to exist the condition is [8, p. 40]

$$a > bA^2, A \text{ being the amplitude.} \tag{35}$$

The exact periodic solution is given by [8, p. 40]

$$x = A\, Sn \left\{ t\, (a - \tfrac{1}{2}\, b\, A^2)^{\frac{1}{2}} \right\} \tag{36}$$

where $Sn \left\{ t\, (a - \frac{1}{2}\, b\, A^2)^{\frac{1}{2}} \right\}$ is the Jacobian elliptic function of modulus $k$ given by

$$k^2 = \frac{bA^2}{(2a - b\, A^2)} \tag{37}$$

and the exact period is given by

$$T = - \frac{4F\, (k, \frac{1}{2}\, \pi)}{(a - \frac{1}{2}\, b\, A^2)^{\frac{1}{2}}} \tag{38}$$

The approximate solution subject to the same initial conditions is given by

$$x_* = \frac{(\beta - \alpha)\, A}{2 + \alpha + \beta} \left( 1 - \frac{\omega_1^2}{\omega_*{}^2} \right) + \left[ 1 + \frac{\alpha - \beta}{2 + \alpha + \beta} \left( 1 - \frac{\omega_1^2}{\omega_*{}^2} \right) \right] A \sin\, (\omega_*\, t + \phi) \tag{39}$$

where

$$\phi = \sin^{-1} \frac{(\alpha - \beta)}{\left[ (\alpha - \beta) + \frac{\omega_*{}^2}{(\omega_*{}^2 - \omega_1{}^2)} (2 + \alpha + \beta) \right]} \tag{40}$$

and

$$\left.\begin{aligned} \omega_1 &= \left[ a - \frac{b\, A^2 \left\{ (\alpha - \beta)^2 + 3\, (\alpha + \beta) + 8 \right\}}{(\alpha + \beta + 3)\, (\alpha + \beta + 4)} \right]^{\frac{1}{2}} \\ \text{and} \quad \omega_* &= \left[ a - \frac{3\, b\, A^2 \left\{ (\alpha - \beta)^2 + \alpha + \beta + 4 \right\}}{(\alpha + \beta + 4)\, (\alpha + \beta + 5)} \right]^{\frac{1}{2}} \end{aligned}\right\} \tag{41}$$

and

$$T_* = 2\pi \left[ a - \frac{3\, b\, A^2 \left\{ (\alpha - \beta)^2 + \alpha + \beta + 4 \right\}}{(4 + \alpha + \beta)\, (5 + \alpha + \beta)} \right]^{-\frac{1}{2}} \tag{42}$$

The limiting behaviour of $T$ and $T_*$ as $A$ (or $\nu$) $\rightarrow 0$ is found to be similar. As $\nu \rightarrow 1$ and 1·29 respectively, both $T$ and $T_* \rightarrow \infty$. For this case $\nu$ must be less than 1 for periodic motion.

Considering the numerical values of the parameters as, $a = 10$, $b = 1$ and $A = 1$ we find that

$$T = 2{\cdot}066. \tag{43}$$

Taking $\alpha = -0{\cdot}45$ and $\beta = -0{\cdot}55$, we find that the approximate period is

$$T_* = 2{\cdot}065 \tag{44}$$

A graphical comparison of the variation of the exact and approximate dimensionless period $T/T'$ with the nonlinearity factor $\nu$ is exhibited in fig. 3.

VARIATION OF THE DIMENSIONLESS PERIOD T/τ WITH NONLINEARITY FACTOR FOR THE DIFFERENTIAL EQUATION
$\ddot{x} + 10x - x^3 = 0$

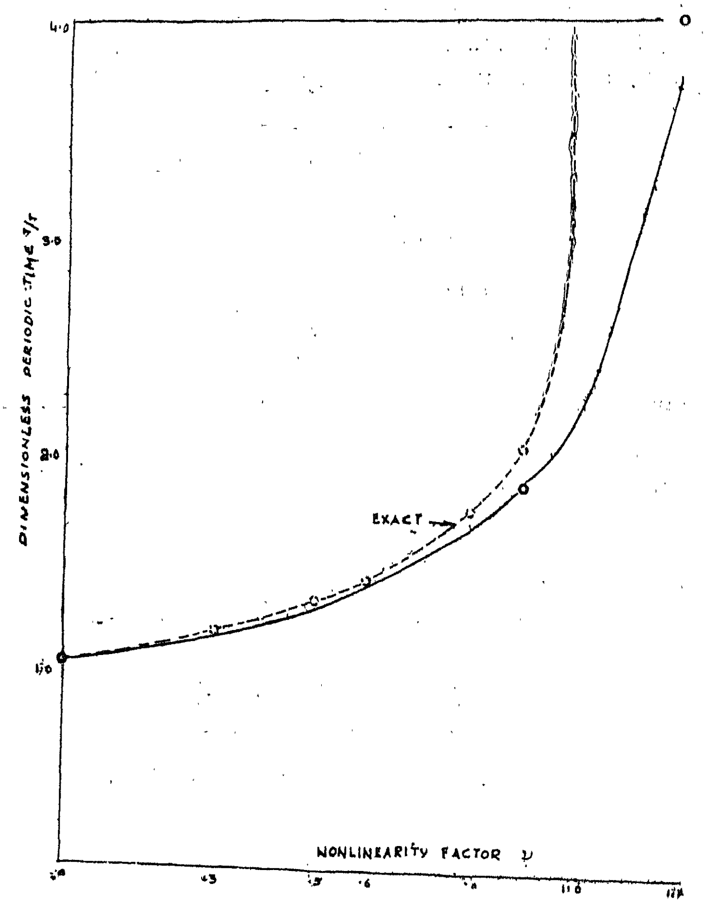



FIG 3

**Case 3.** *Softening-Hardening cubic* : In this case

$$f(x) = -ax + bx^3, \ (a, b > 0)$$

The exact solution in terms of elliptic functions subject to the conditions $x = A$, $\dot{x} = 0$ at $t = 0$, is

$$x = A \, dn \{ At (b/2)^{\frac{1}{2}} \} \tag{45}$$

and the modulous is given by

$$k^2 = 2(1 - a/bA^2). \tag{46}$$

and the exact period is

$$T = \frac{2\,F\,(k, \frac{1}{2}\,\pi)}{A(b/2)^{\frac{1}{2}}} \tag{47}$$

The singular points of the differential equation are $(0, 0)$, $(\pm\sqrt{a/b}, 0)$, of which the later two are centres, so that a stable periodic motion exists about each one of them. The solution represented by (45) is the stable periodic solution about $(\sqrt{a/b}, 0)$. In order to obtain the approximate period solution about the same point, we change the origin to the point $(\sqrt{a/b}, 0)$ so that the transformed differential equation becomes

$$\ddot{x} + 2a\,x + 3\sqrt{ab}\,x^2 + b\,x^3 = 0 \tag{48}$$

Linearising this equation with the help of Jacobi polynomials in the interval $(-A, A)$, we get

$$\ddot{x} + \omega_*^2\,x = -\frac{(\alpha-\beta)A}{(2+\alpha+\beta)}\,(\omega_*^2 - \omega_1^2) \tag{49}$$

where $\omega_1^2 = 2a + \dfrac{3\sqrt{ab}\;A\,\{(\beta-\alpha)^2 + (\alpha+\beta) + 2\}}{(\alpha+\beta+3)\,(\beta-\alpha)} + \dfrac{bA^2\,\{(\alpha-\beta)^2 + 3(\alpha+\beta) + 8\}}{(\alpha+\beta+3)\,(\alpha+\beta+4)}$

$$\omega_*^2 = 2a + \frac{3\sqrt{ab}\,A\,(\beta-\alpha)}{4+\alpha+\beta} + \frac{3b\,A^2\,[(\alpha-\beta)^2 + \alpha+\beta+4]}{(5+\alpha+\beta)\,(4+\alpha+\beta)} \tag{50}$$

The approximate solution is thus given by (subject to the same initial conditions)

$$x_* = \left[A + \frac{(\alpha-\beta)A}{2+\alpha+\beta}\left(\frac{\omega_*^2 - \omega_1^2}{\omega_*^2}\right)\right]\cos\omega_* t + \frac{(\beta-\alpha)A}{2+\alpha+\beta}\left(\frac{\omega_*^2 - \omega_1^2}{\omega_*^2}\right) \tag{51}$$

Considering numerical values as

$a = 10$, $b = 15$, $A = 1$, we find that

$$T = 1{\cdot}48 \tag{52}$$

Choosing the values of $\alpha$ and $\beta$ as $\alpha = 1{\cdot}0$ and $\beta = -0{\cdot}5$, we have the approximate period given by

$$T_* = 1{\cdot}46 \tag{53}$$

(*ii*) *Sine nonlinearity* : In this case let $f(x) = \omega_*^2 \sin x$ so that the differential equation is

$$\ddot{x} + \omega_0^2 \sin x = 0 \tag{54}$$

Approximating $\sin x$ in the interval $(-A, A)$ by means of linear Jacobi polynomials, one obtains

$$(\omega_0^2 \sin x)_* = a_0^{(\alpha,\beta)}\,P_0^{(\alpha,\beta)}(x/A) + a_1^{(\alpha,\beta)}\,P_1^{(\alpha,\beta)}(x/A) \tag{55}$$

where $a_0^{(\alpha,\beta)} = \dfrac{\Gamma(2+\alpha+\beta)}{2^{1+\alpha+\beta}\,\Gamma(1+\alpha)\,\Gamma(1+\beta)}\displaystyle\int_{-1}^{1} \omega_0^2 \sin Ax\,(1-x)^\alpha\,(1+x)^\beta\,P_0^{(\alpha,\beta)}(x)\,dx$

$$\text{and}\quad a_1^{(\alpha,\beta)} = \frac{\int_{-1}^{1} \omega_0^2 \sin Ax\,(1-x)^\alpha\,(1+x)^\beta\,P_1^{(\alpha,\beta)}(x)\,dx}{g_1} \tag{56}$$

After a little manipulation one finds that

$$a_0^{(\alpha,\beta)} = \left(\frac{\alpha-\beta}{\alpha+\beta+2}\right) A\,\omega_0^2 \sum_{s=0}^{\infty} \frac{\Gamma(2s+2+\beta)\,\Gamma(3+\alpha+\beta)}{(\alpha-\beta)\,\Gamma(1+\beta)\,\Gamma(2s+\alpha+\beta+3)} \frac{(-1)^s A^{2s}}{\lfloor 2s+1}$$

$$\times\ 2F_1\left[\begin{matrix} -2s-1,\ 1+\alpha\ ; \\ -\beta-2s-1\ ; \end{matrix}\ -1\right] \tag{57}$$

and $$a_1^{(\alpha,\beta)} = \left(\frac{\alpha-\beta}{\alpha+\beta+2}\right) A\,\omega_0^2 \sum_{s=0}^{\infty} \frac{\Gamma(2s+2+\beta)\,\Gamma(4+\alpha+\beta)}{\Gamma(2+\beta)\,\Gamma(2s+4+\alpha+\beta)} \frac{(-1)^s A^{2s}}{\lfloor 2s}$$

$$\times\ 2\,F_1\left[\begin{matrix} -2s,\ 2+\alpha\ ; \\ -\beta-2s-1\ ; \end{matrix}\ -1\right] \tag{57}$$

Hence (54) becomes

$$(\omega_0^2 \sin x)_* = \omega_*^2 x + \frac{(\alpha-\beta)A}{(2+\alpha+\beta)} (\omega_*^2 + \omega_1^2) \tag{58}$$

where

$$\omega_1^2 = \omega_0^2 \sum_{s=0}^{\infty} \frac{\Gamma(2s+2+\beta)\,\Gamma(3+\alpha+\beta)}{\Gamma(2+\beta)\,\Gamma(2s+3+\alpha+\beta)} \frac{(-1)^s A^{2s}}{(\alpha-\beta)\,\lfloor 2s+1}\ 2F_1\left[\begin{matrix} -2s-1,\ 1+\alpha\ ; \\ -2s-\beta-1\ , \end{matrix}\ -1\right]$$

$$\omega_*^2 = \omega_0^2 \sum_{s=0}^{\infty} \frac{\Gamma(2s+2+\beta)\,\Gamma(4+\alpha+\beta)}{\Gamma(2+\beta)\,\Gamma(2s+4+\alpha+\beta)} \frac{(-1)^s A^{2s}}{2s\,!}\ 2F_1\left[\begin{matrix} -2s,\ 2+\alpha\ ; \\ -2s-1-\beta\ ; \end{matrix}\ -1\right] \tag{59}$$

The differential equation thus reduces to

$$\ddot{x} + \omega_*^2 x = \frac{(\beta-\alpha)A}{2+\alpha+\beta} (\omega_*^2 + \omega_1^2) \tag{60}$$

The approximate solution subject to the initial conditions $x = A$ and $\dot{x} = 0$ at $t = 0$ is thus

$$x_* = \left[ A - \frac{(\beta-\alpha)}{2+\alpha+\beta} A \left(\frac{\omega_*^2 + \omega_1^2}{\omega_*^2}\right)\right] \cos \omega_* t + \frac{(\beta-\alpha)A}{2+\alpha+\beta}\left(\frac{\omega_*^2 + \omega_1^2}{\omega_*^2}\right) \tag{61}$$

and the approximate periodic time is given by

$$T_* = 2\pi/\omega_* \tag{62}$$

The exact period is given by [7, p. 33]

$$T = \frac{4F(k, \tfrac{1}{2}\pi)}{\omega_0} \tag{63}$$

where $F(k, \frac{1}{2}\pi)$ is the complete elliptic integral of the first kind of modulus

$$k = \sin A/2 \tag{64}$$

when $A = \frac{1}{2}$, we have

$$T = 6{\cdot}40/\omega_0 \tag{65}$$

Choosing $\alpha = -0{\cdot}4$ and $\beta = -0{\cdot}6$, we find that

$$\omega_*^2 = 0{\cdot}9688\,\omega_0^2 \tag{66}$$

Hence

$$T_* = \frac{6{\cdot}48}{\omega_0} \tag{67}$$

For oscillatory motion it is necessary that

$$|A| < \pi \quad \text{since } |k| < 1$$

The limiting behaviour of $T_*$ as $A \to 0$ agrees with that of $T$.

*(iii) Hyperbolic sine nonlinearity*: For this case let $f(x) = \omega_0^2 \sinh x$ so that the differential equation is

$$\ddot{x} + \omega_0^2 \sinh x = 0 \tag{68}$$

The exact solution of this equation is known in terms of elliptic functions and the exact period is given by [4, p. 279]

$$T = 4/\omega_0 \operatorname{sech} A/2 \, F(k, \tfrac{1}{2}\pi) \tag{69}$$

where $A$ is the amplitude of the motion and

$$k = \tanh A/2$$

Approximating $\sinh x$ in the interval $(-A, A)$ by means of linear Jacobi polynomials one obtains

$$(\omega_0^2 \sinh)x_* = \omega_*^2 x + \frac{(\alpha - \beta) A}{2 + \alpha + \beta} (\omega_*^2 + \omega_1^2)$$

where

$$\omega_1^2 = \omega_0^2 \sum_{s=0}^{\infty} \frac{\Gamma(2s + 2 + \beta)\Gamma(3 + \alpha + \beta)\, A^{2s}}{\Gamma(2 + \beta)\Gamma(2s + 3 + \alpha + \beta)\;(\alpha - \beta)\,|2s+1|}\, {}_2F_1 \begin{bmatrix} -2s-1,\ 1+\alpha\ ; & -1 \\ -2s-\beta-1\ ; & \end{bmatrix}$$

$$\text{and } \omega_*^2 = \omega_0^2 \sum_{s=0}^{\infty} \frac{\Gamma(2s + 2 + \beta)\Gamma(4 + \alpha + \beta)}{\Gamma(2 + \beta)\Gamma(2s + 4 + \alpha + \beta)} \frac{A^{2s}}{2s\,!}\, {}_2F_1 \begin{bmatrix} -2s,\ 2+\alpha\ ; & -1 \\ -2s-1-\beta\ ; & \end{bmatrix} \tag{70}$$

Thus the differential equation reduces to

$$\ddot{x} + \omega_*^2 x = \frac{(\beta - \alpha)A}{(2 + \alpha + \beta)} (\omega_*^2 + \omega_1^2)$$

The approximate period of oscillation is thus

$$T_* = \frac{2\pi}{\omega_*} \tag{71}$$

Considering numerical values as :

$$A = \tfrac{1}{2},\ \alpha = -0{\cdot}4 \text{ and } \beta = -0{\cdot}6, \text{ we have}$$

$$T_* = \frac{6{\cdot}085}{\omega_0} \tag{72}$$

As $A \to 0$, both $T$ and $T_*$ approach to $2\pi/\omega_0$

**Discussion of results**: The exact solution for the case 1 discussed in section 5(*i*) is plotted in figure 1, along with the approximate solution corresponding to $\alpha = -0{\cdot}4$ and $\beta = -0{\cdot}5$. The variation of the dimensionless periodic time with the nonlinearity factor for the exact and approximate case are plotted in figure 2. In both the figures there is a close agreement in the exact and approximate results. Further a comparison with figure 1[3] shows that the Jacobi polynomial approximation is closer than the Gegenbauer polynomial approximation used in[3].

Figure 3 shows that the variation of the dimensionless exact period for case 2 with the nonlinearity factor along with the variation of the approximate

dimensionless period corresponding to $\alpha = -0\cdot45$ and $\beta = -0\cdot55$. The approximate curve agrees closely with the exact curve.

The exact periodic time for the case 3 section 5($i$) agrees closely with the approximate period corresponding to $\alpha = 1\cdot0$ and $\beta = -0\cdot5$.

In the case of the differential equation discussed in section 5($ii$) the approximate periodic time corresponding to $\alpha = -0\cdot4$ and $\beta = -0\cdot6$ is found to agree closely with the exact period. Same fact is observed for the differential equation discussed in section 5($iii$).

**7. Conclusions :** In the present paper the linearisation of the nonlinear ordinary differential equation has been accomplished by the linear Jacobi polynomial approximation of the nonlinear restoring force or torque. The results thus obtained are found to agree closely with the exact results which can be obtained by means of elliptic functions. The approximation can be made very close by a suitable choice of $\alpha$ and $\beta$. The Jacobi polynomial approximation has an advantage over the ultraspherical and Gegenbauer polynomial approximation used in[1,2] and [3,4] that the average error over a quarter period is much less in case of Jacobi polynomial approximation as compared to others. In case of the ultraspherical and Gegenbauer polynomial approximation, the approximate solution curve is entirely on one side of the exact solution, where as in case of Jacobi polynomial approximation, it is partly on one side and partly on the other amounting to a rotation of the curve. It may be noted that results obtained in this paper reduce into corresponding results obtained in[1] and[2] by putting $\alpha = \beta = \lambda - \frac{1}{2}$

The differential equations discussed in section 5 occur in several physical problems like the motion of a mass on nonlinear spring, motion of a simple pendulum, the elastica, electrical circuit with nonlinear capacitor etc.

**Acknowledgements**

The author is highly thankful to Dr. B. R. Bhonsle, Professor of Applied Mathematics, Government Engineering College, Jabalpur for suggesting the problem and for his guidance and valuable suggestions. My thanks are also due to Principal A. L. Swamy for providing me facilities to carry out the work.

**References**

1. Denman, H. H. and Howard, J. E. Applications of ultraspherical polynomials to nonlinear oscillations—I Free oscillations of the pendulum. *Q. Appl. Math.* 21 : 325–330, (1964).
2. Denman, H. H. and Liu, Y. K. Applications of ultraspherical polynomials to nonlinear oscillations II-Free oscillations, *Q. App. Math.*, 22 : 273–292, (1964).
3. Garde, R. M. Applications of Gegenbauer polynomials to nonlinear oscillations—Forced and Free oscillations without damping, the Indian Journal of Mathematics, 7 (2) : 111–117, (1965).
4. Garde, R. M. Application of Gegenbauer polynomials to nonlinear damped oscillations, communicated for publication to the Journal of Science and Engineering Research.
5. Rainville, E. D. Special functions, *McMillan Co., New York*, (1960).
6. Varma, R. C. On some integrals involving Jacobi polynomials, to be published in National Academy of Sciences, India.
7. Bhonsle, B. R. On some results involving Jacobi polynomials, Bulletin of the Calcutta Mathematical Society, 50 (3) : 160, (1958).
8. McLachlan, N. W. Ordinary nonlinear differential equations for engineers and physicists second edition.